

Portfolio Theory

Chapter 1: Portfolio Theory

Agenda

- 1 Portfolio Theory
 - Risky Assets Only
 - Risky and Risk-free Assets

- 2 Tangency Portfolio

- 3 Fund Theorems

- 4 No-short-sell Constraints

- 5 Estimation of Parameters

- 6 Real Data Application

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Portfolio Theory: Settings

- We consider n assets over a **single period** of time $[t, t + 1]$
- For the i -th asset ($i \in \{1, \dots, n\}$), denote:
 - $P_i(t)$ = Price of asset i at time t
 - R_i = Return or log-return = $\frac{P_i(t+1)}{P_i(t)} - 1$ or $\log \frac{P_i(t+1)}{P_i(t)}$
 - By $\log(1 + x) \approx x$, we have $\log \frac{P_i(t+1)}{P_i(t)} \approx \frac{P_i(t+1)}{P_i(t)} - 1$
 - $\mu_i = E(R_i)$ = Expected return
 - $\sigma_i^2 = \text{Var}(R_i)$ = Variance of return (**a measure of risk**)
 - $\sigma_{ij} = \text{Cov}(R_i, R_j)$ = Covariance of returns of assets i and j
- Goal:
 - Assign **weights** $\mathbf{w} = (w_1, \dots, w_n)^T$ on the assets to minimize portfolio risk for a given target of expected return μ_* :

$$\min_{\mathbf{w} \in \mathbb{R}^n} \text{Var} \left(\sum_{i=1}^n w_i R_i \right) \quad \text{subject to} \quad E \left(\sum_{i=1}^n w_i R_i \right) = \mu_* \quad \text{and} \quad \sum_{i=1}^n w_i = 1$$

Risk Minimization

$$\text{MVP: } \min_{\mathbf{w} \in \mathbb{R}^n} \text{Var} \left(\sum_{i=1}^n w_i R_i \right) \quad \text{subject to} \quad E \left(\sum_{i=1}^n w_i R_i \right) = \mu_* \text{ and } \sum_{i=1}^n w_i = 1$$

Using the Lagrange multiplier method, we have to solve

$$\min_{\mathbf{w} \in \mathbb{R}^n} \mathbf{w}^T \Sigma \mathbf{w} - 2\lambda_\mu (\mathbf{w}^T \boldsymbol{\mu} - \mu_*) - 2\lambda_1 (\mathbf{w}^T \mathbf{1} - 1),$$

- $\mathbf{w} = (w_1, \dots, w_n)^T$ (portfolio weights)
- $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$ (expected assets returns)
- $\Sigma = (\text{Cov}(R_i, R_j))_{1 \leq i, j \leq n} = (\sigma_{ij})_{1 \leq i, j \leq n}$ (covariance of assets returns)
- $\text{Var} \left(\sum_{i=1}^n w_i R_i \right) = \text{Cov} \left(\sum_{i=1}^n w_i R_i, \sum_{j=1}^n w_j R_j \right) = \sum_{i,j=1}^n w_i w_j \sigma_{ij} = \mathbf{w}^T \Sigma \mathbf{w}$
- $E \left(\sum_{i=1}^n w_i R_i \right) = \sum_{i=1}^n w_i E(R_i) = \sum_{i=1}^n w_i \mu_i = \mathbf{w}^T \boldsymbol{\mu}$
- $\sum_{i=1}^n w_i = \mathbf{w}^T \mathbf{1}$

Remarks:

- The constant 2 at the Lagrange multipliers is added to simplify calculations
- Equivalent to $\min_{\mathbf{w} \in \mathbb{R}^n} \frac{1}{2} \mathbf{w}^T \Sigma \mathbf{w} - \lambda_\mu (\mathbf{w}^T \boldsymbol{\mu} - \mu_*) - \lambda_1 (\mathbf{w}^T \mathbf{1} - 1),$
- Maximizing $\frac{1}{2} \mathbf{w}^T \Sigma \mathbf{w}$ and $\mathbf{w}^T \Sigma \mathbf{w}$ are the same

Markowitz Portfolio Theory



A good portfolio is more than a long list of good stocks and bonds. It is a balanced whole, providing the investor with protections and opportunities with respect to a wide range of contingencies.

— *Harry Markowitz* —

AZ QUOTES

Harry Markowitz (1927~2023): Nobel Economics 1990

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Minimum Variance Portfolio

Definition 1 (Minimum Variance Portfolio (MVP))

- If asset return has expectation $\boldsymbol{\mu} \in \mathbb{R}^n$ and covariance $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$,
- the **Minimum Variance Portfolio (MVP)** is an optimal portfolio that
 - allocate weights $\mathbf{w}_* = (w_{*1}, \dots, w_{*n})$ on the n assets
 - satisfies $\mathbf{w}_* = \arg \min_{\mathbf{w}} \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$ subject to $\mathbf{w}^T \boldsymbol{\mu} = \mu_*$ and $\mathbf{w}^T \mathbf{1} = 1$

Theorem 1 (The Minimum Variance Portfolio Composition)

If $\boldsymbol{\Sigma}$ is invertible, then the solution of MVP is

- $\mathbf{w}_*(\mu_*) = \Delta^{-1}(c_{11}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - c_{1\mu}\boldsymbol{\Sigma}^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\boldsymbol{\Sigma}^{-1}\mathbf{1} - c_{1\mu}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})$,
 - $c_{11} = \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}$
 - $c_{1\mu} = \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}$
 - $c_{\mu\mu} = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$
 - $\Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$

- If risk-free asset is present, then covariance matrix $\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma} \end{pmatrix}$ is not invertible.
- $\boldsymbol{\Sigma}$ is not invertible $\Rightarrow$ some asset is a linear combination of other assets
 $\Rightarrow$ can remove redundant assets

Minimum Variance Portfolio

Thm 1: $\min_{\mathbf{w} \in \mathbb{R}^n} \text{Var} \left(\sum_{i=1}^n w_i R_i \right)$ subject to $E \left(\sum_{i=1}^n w_i R_i \right) = \mu_*$ & $\sum_{i=1}^n w_i = 1$

MVP: $\mathbf{w}_*(\mu_*) = \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})$

• $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}, \quad c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}, \quad c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}, \quad \Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$

Proof.

Lagrangian: $L(\mathbf{w}, \lambda_\mu, \lambda_1) = \mathbf{w}^T \Sigma \mathbf{w} - 2\lambda_\mu(\mathbf{w}^T \boldsymbol{\mu} - \mu_*) - 2\lambda_1(\mathbf{w}^T \mathbf{1} - 1)$

• Recall $\frac{\partial \mathbf{c}^T \mathbf{x}}{\partial \mathbf{x}} = \mathbf{c}$, $\frac{\partial \mathbf{x}^T \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = \mathbf{A}^T \mathbf{x} + \mathbf{A} \mathbf{x}$, and $\Sigma^T = \Sigma$:

$$\frac{\partial L}{\partial \mathbf{w}} = 2(\Sigma \mathbf{w} - \lambda_\mu \boldsymbol{\mu} - \lambda_1 \mathbf{1}) = \mathbf{0} \quad (1)$$

$$\frac{\partial L}{\partial \lambda_\mu} = -2(\mathbf{w}^T \boldsymbol{\mu} - \mu_*) = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda_1} = -2(\mathbf{w}^T \mathbf{1} - 1) = 0 \quad (3)$$

• Solving the system of equations:

$$\boldsymbol{\mu}^T \Sigma^{-1} \times (1) : \quad \boldsymbol{\mu}^T \mathbf{w} - \lambda_\mu \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu} - \lambda_1 \boldsymbol{\mu}^T \Sigma^{-1} \mathbf{1} \stackrel{(2)}{=} \mu_* - c_{\mu\mu} \lambda_\mu - c_{1\mu} \lambda_1 = 0$$

$$\mathbf{1}^T \Sigma^{-1} \times (1) : \quad \mathbf{1}^T \mathbf{w} - \lambda_\mu \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu} - \lambda_1 \mathbf{1}^T \Sigma^{-1} \mathbf{1} \stackrel{(3)}{=} 1 - c_{1\mu} \lambda_\mu - c_{11} \lambda_1 = 0$$

$$\Rightarrow \quad \lambda_\mu = (\mu_* c_{11} - c_{1\mu}) / \Delta, \quad \lambda_1 = (c_{\mu\mu} - c_{1\mu} \mu_*) / \Delta, \quad \text{where } \Delta = c_{\mu\mu} c_{11} - c_{1\mu}^2$$

• Sub λ_1, λ_μ back to (1) $\Rightarrow \mathbf{w} = \lambda_\mu \Sigma^{-1} \boldsymbol{\mu} + \lambda_1 \Sigma^{-1} \mathbf{1} =: \mathbf{w}_*$

MVP Example

$$\text{MVP: } \mathbf{w}_*(\mu_*) = \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})$$

• $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}, \quad c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}, \quad c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}, \quad \Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$

Example 1

Find the optimal portfolio for $\mu_* = 1.5\%$, given assets characteristics below:

$$\boldsymbol{\mu} = \begin{pmatrix} 2\% \\ 1\% \\ 1.5\% \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.25\% & 0.10\% & 0.075\% \\ 0.10\% & 0.16\% & 0.12\% \\ 0.075\% & 0.12\% & 0.36\% \end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix} 533.33 & -333.33 & 0 \\ -333.33 & 1041.67 & -277.78 \\ 0 & -277.78 & 370.37 \end{pmatrix}$$

• $\Sigma^{-1}\mathbf{1} = (200, 430.5556, 92.5926)^T; \quad \Sigma^{-1}\boldsymbol{\mu} = (7.3333, -0.4167, 2.7778)^T$

• $c_{11} = 200 + 430.5556 + 92.5926 = 723.1481$

• $c_{1\mu} = 7.3333 - 0.4167 + 2.7778 = 9.6944$

• $c_{\mu\mu} = (2\%)(7.3333) + (1\%)(-0.4167) + (1.5\%)(2.7778) = 0.1842$

• $\Delta = (723.1481)(0.1842) - (9.6944)^2 = 39.1975$

• $\mathbf{w}_* = \frac{1}{39} \left[723 \begin{pmatrix} 7.33 \\ -0.42 \\ 2.78 \end{pmatrix} - 9.69 \begin{pmatrix} 200 \\ 431 \\ 92.6 \end{pmatrix} \right] 1.5\% + \frac{1}{39} \left[0.18 \begin{pmatrix} 200 \\ 431 \\ 92.6 \end{pmatrix} - 9.69 \begin{pmatrix} 7.33 \\ -0.42 \\ 2.78 \end{pmatrix} \right]$
 $= (41.34\%, 41.34\%, 17.32\%)^T$

• $\sigma = \sqrt{\mathbf{w}_*^T \Sigma \mathbf{w}_*} \approx 3.78\%$

Efficient Frontier

$$\text{MVP: } \mathbf{w}_*(\mu_*) = \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})$$

$$\bullet \quad c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}, \quad c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}, \quad c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}, \quad \Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$$

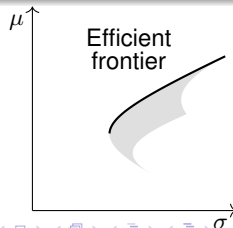
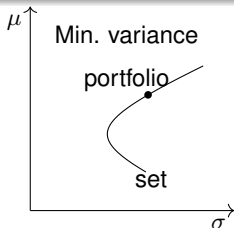
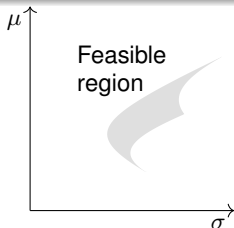
- For a target return μ_* , the optimal weight $\mathbf{w}_* = \mathbf{w}_*(\mu_*)$ has a variance

$$\sigma_*^2(\mu_*) = \mathbf{w}_*^T \Sigma \mathbf{w}_*$$

- The curve $(\sigma_*(\mu_*), \mu_*)$ w.r.t. μ_* is the **minimum variance set (MVS)**.
- The **efficient frontier** is the **upper part** of the curve $(\sigma_*(\mu_*), \mu_*)$

Theorem 2 (Geometry of the Minimum Variance Set)

The MVS is a **hyperbola**: $\sigma_*^2(\mu_*) = \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu})$



Theorem 2: $\sigma_*^2(\mu_*) = \mathbf{w}_*^T \Sigma \mathbf{w}_* = \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu})$

Proof.

• Recall

- (1): $\Sigma \mathbf{w}_* - \lambda_\mu \boldsymbol{\mu} - \lambda_1 \mathbf{1} = \mathbf{0}$
- (2): $\mathbf{w}_*^T \boldsymbol{\mu} = \mu_*$
- (3): $\mathbf{w}_*^T \mathbf{1} = 1$
- $\lambda_\mu = (\mu_* c_{11} - c_{1\mu}) / \Delta$
- $\lambda_1 = (c_{\mu\mu} - c_{1\mu} \mu_*) / \Delta$
- $\Delta = c_{\mu\mu} c_{11} - c_{1\mu}^2$

• Find σ_*^2 :

$$\begin{aligned} \mathbf{w}_*^T \times (1) : \mathbf{w}_*^T \Sigma \mathbf{w}_* &= \lambda_\mu \mathbf{w}_*^T \boldsymbol{\mu} + \lambda_1 \mathbf{w}_*^T \mathbf{1} \\ \xRightarrow{(2)(3)} \sigma_*^2 &= \Delta^{-1} ((\mu_* c_{11} - c_{1\mu}) \mu_* + (c_{\mu\mu} - c_{1\mu} \mu_*)) \\ &= \Delta^{-1} (c_{11} \mu_*^2 - 2c_{1\mu} \mu_* + c_{\mu\mu}) \end{aligned}$$



Efficient Frontier

- **MVP:** $\mathbf{w}_*(\mu_*) = \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})$
- **Theorem 2:** $\sigma_*^2(\mu_*) = \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu})$
- $c_{11} = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$, $c_{1\mu} = \mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}$, $c_{\mu\mu} = \boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}$, $\Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$

Corollary 2 (Overall Minimal Variance Portfolio (OMVP))

If the asset return has expectation $\boldsymbol{\mu} \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n \times n}$, then the portfolio that achieves the minimal risk is given by

$$\mathbf{w}_{\min\text{var}} = \Sigma^{-1}\mathbf{1}/c_{11}$$

Proof.

- From **Theorem 2**, the minimum portfolio variance for a fixed return μ_* is

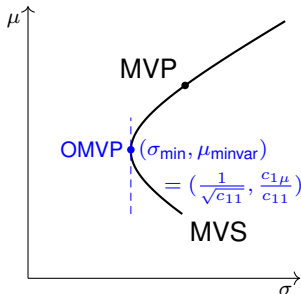
$$\begin{aligned}\sigma_*^2(\mu_*) &= \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu}) \\ &= \Delta^{-1}\left[c_{11}\left(\mu_* - \frac{c_{1\mu}}{c_{11}}\right)^2 + c_{\mu\mu} - \frac{c_{1\mu}^2}{c_{11}}\right]\end{aligned}$$

- As $\sigma_*^2(\mu_*)$ is minimum (equals $\sigma_{\min}^2 := 1/c_{11}$) when $\mu_* = \mu_{\min\text{var}} := c_{1\mu}/c_{11}$,

$$\begin{aligned}\mathbf{w}_{\min\text{var}} &= \mathbf{w}_*\left(\frac{c_{1\mu}}{c_{11}}\right) = \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\frac{c_{1\mu}}{c_{11}} + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu}) \\ &= \Delta^{-1}\left(c_{\mu\mu} - c_{1\mu}\frac{c_{1\mu}}{c_{11}}\right)\Sigma^{-1}\mathbf{1} = \Sigma^{-1}\mathbf{1}/c_{11}\end{aligned}$$

Efficient Frontier

- **Theorem 2:** $\sigma_*^2(\mu_*) = \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu}), \quad \Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$
- $\mathbf{w}_{\min\text{var}} = \Sigma^{-1}\mathbf{1}/c_{11}$
- $\mu_{\min\text{var}} = \frac{c_{1\mu}}{c_{11}}, \quad \sigma_{\min}^2 = \sigma_*^2(\mu_{\min\text{var}}) = \Delta^{-1}(c_{1\mu}^2/c_{11} - 2c_{1\mu}^2/c_{11} + c_{\mu\mu}) = \frac{1}{c_{11}}$



Example 2

Recall Example 1:

$$c_{11} = 723.1481, c_{1\mu} = 9.6944, c_{\mu\mu} = 0.1842, \Delta = 39.1975, \Sigma^{-1}\mathbf{1} = (200, 430.5556, 92.5926)^T$$

$$\text{MVS: } \sigma_*^2(\mu_*) = 18.4488\mu_*^2 - 0.4947\mu_* + 0.004698$$

$$\Rightarrow \mathbf{w}_{\min\text{var}} = (0.2766, 0.5954, 0.1280)^T, \quad \mu_{\min\text{var}} = 1.3406\%, \quad \sigma_{\min}^2 = 0.1383\%, \quad \sigma_{\min} = 3.72\%$$

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Portfolio including Risk-free and Risky Assets

Minimum Variance Portfolio (MVP) is an optimal portfolio that satisfies

- $\mathbf{w}_* = \arg \min_{\mathbf{w}} \mathbf{w}^T \Sigma \mathbf{w}$ subject to $\mathbf{w}^T \boldsymbol{\mu} = \mu_*$ and $\mathbf{w}^T \mathbf{1} = 1$
- If Σ^{-1} exists, then
$$\mathbf{w}_*(\mu_*) = \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})$$

When a risk-free asset (e.g. U.S. Treasury Bill) with return R_f is included,

- Expected return = $E(R_f) = r$ = Risk-free rate
- Risk = Variance of return = $\text{Var}(R_f) = 0$
- Covariance $\text{Cov}(R_f, R_i) = 0$ for all $i = 1, \dots, n$
- The covariance matrix of $(R_f, R_1, \dots, R_n)$ is defined as

$$\tilde{\Sigma} = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \Sigma \end{pmatrix}$$

where $\Sigma = (\text{Cov}(R_i, R_j))_{1 \leq j \leq n} = (\sigma_{ij})_{1 \leq j \leq n}$

- As $\tilde{\Sigma}^{-1}$ does not exist, we need to modify the derivation of $\mathbf{w}_*$

Portfolio including Risk-free and Risky Assets

Theorem 3 (The MVP composition with Risk-free Assets)

- If asset return has expectation $\tilde{\boldsymbol{\mu}} = \begin{pmatrix} r \\ \boldsymbol{\mu} \end{pmatrix} \in \mathbb{R}^{n+1}$ and covariance

$$\tilde{\boldsymbol{\Sigma}} = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \boldsymbol{\Sigma} \end{pmatrix} \in \mathbb{R}^{(n+1) \times (n+1)},$$

with $\boldsymbol{\Sigma}^{-1}$ exists,

- then the **Minimum Variance Portfolio (MVP)** $\tilde{\mathbf{w}}_* \in \mathbb{R}^{n+1}$ for a target return μ_* is given by

$$\begin{aligned} \tilde{\mathbf{w}}_* &= \arg \min_{\tilde{\mathbf{w}}} \tilde{\mathbf{w}}^T \tilde{\boldsymbol{\Sigma}} \tilde{\mathbf{w}} \text{ subject to } \tilde{\mathbf{w}}^T \tilde{\boldsymbol{\mu}} = \mu_* \text{ and } \tilde{\mathbf{w}}^T \mathbf{1} = 1 \\ &= (w_{r*}, \mathbf{w}_*^T)^T, \end{aligned}$$

where

- $w_{r*} = (\boldsymbol{\mu} - \mu_* \mathbf{1})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r \mathbf{1}) \gamma^{-2} \in \mathbb{R}$
- $\mathbf{w}_* = (\mu_* - r) \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r \mathbf{1}) \gamma^{-2} \in \mathbb{R}^n$
- $\gamma^2 = (\boldsymbol{\mu} - r \mathbf{1})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r \mathbf{1}) = c_{\mu\mu} - 2rc_{1\mu} + r^2 c_{11}$

Portfolio including Risk-free and Risky Assets

Thm 3: MVP: $\tilde{\mathbf{w}}_* = \arg \min_{\tilde{\mathbf{w}}} \tilde{\mathbf{w}}^T \tilde{\Sigma} \tilde{\mathbf{w}}$ subject to $\tilde{\mathbf{w}}^T \tilde{\boldsymbol{\mu}} = \mu_*$ and $\tilde{\mathbf{w}}^T \mathbf{1} = 1$.

$\Rightarrow \tilde{\mathbf{w}}_* = (w_{r*}, \mathbf{w}_*^T)^T$, where $w_{r*} = \frac{c_{\mu\mu} - (r + \mu_*)c_{1\mu} + r\mu_*c_{11}}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$, $\mathbf{w}_* = \frac{(\mu_* - r)(\Sigma^{-1}\boldsymbol{\mu} - r\Sigma^{-1}\mathbf{1})}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$

• $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$, $c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}$, $c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$, if Σ^{-1} exists.

Proof.

- Let $\tilde{\mathbf{w}}^T = (w_r, w_1, \dots, w_n) = (w_r, \mathbf{w}^T)$
- **Caution:** $\tilde{\mathbf{w}}^T \tilde{\Sigma} \tilde{\mathbf{w}} = \mathbf{w}^T \Sigma \mathbf{w}$ does not contain w_r
- Lagrangian: $L(\tilde{\mathbf{w}}, \lambda_\mu, \lambda_1) = \mathbf{w}^T \Sigma \mathbf{w} - 2\lambda_\mu(\tilde{\mathbf{w}}^T \tilde{\boldsymbol{\mu}} - \mu_*) - 2\lambda_1(\tilde{\mathbf{w}}^T \mathbf{1} - 1)$

$\partial L / \partial w_r$	$\partial L / \partial \mathbf{w}$	$\partial L / \partial \lambda_\mu$	$\partial L / \partial \lambda_1$
$2(-\lambda_\mu r - \lambda_1)$	$2(\Sigma \mathbf{w} - \lambda_\mu \boldsymbol{\mu} - \lambda_1 \mathbf{1})$	$-2(\tilde{\mathbf{w}}^T \tilde{\boldsymbol{\mu}} - \mu_*)$	$-2(\tilde{\mathbf{w}}^T \mathbf{1} - 1)$

- Setting derivatives = 0 gives the system of equations:

(1) $\partial L / \partial w_r$	(2) $\partial L / \partial \mathbf{w}$	(3) $\partial L / \partial \lambda_\mu$	(4) $\partial L / \partial \lambda_1$
$\lambda_\mu r + \lambda_1 = 0$	$\mathbf{w} = \lambda_\mu \Sigma^{-1} \boldsymbol{\mu} + \lambda_1 \Sigma^{-1} \mathbf{1}$	$w_r r + \mathbf{w}^T \boldsymbol{\mu} = \mu_*$	$w_r + \mathbf{w}^T \mathbf{1} = 1$

$$\boldsymbol{\mu}^T \times (2) \stackrel{(3)}{\Rightarrow} \mu_* - rw_r = \lambda_\mu c_{\mu\mu} + \lambda_1 c_{1\mu} \stackrel{(1)}{=} \lambda_\mu (c_{\mu\mu} - c_{1\mu} r)$$

$$\mathbf{1}^T \times (2) \stackrel{(4)}{\Rightarrow} 1 - w_r = \lambda_\mu c_{1\mu} + \lambda_1 c_{11} \stackrel{(1)}{=} \lambda_\mu (c_{1\mu} - c_{11} r)$$

$$\Rightarrow \lambda_\mu = \frac{\mu_* - r}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}, \quad w_{r*} = \frac{c_{\mu\mu} - (r + \mu_*)c_{1\mu} + r\mu_*c_{11}}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}, \quad \mathbf{w}_* = \frac{(\mu_* - r)(\Sigma^{-1}\boldsymbol{\mu} - r\Sigma^{-1}\mathbf{1})}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$$

Portfolio Including Risk-free and Risky Assets

MVP: $\tilde{\mathbf{w}}_* = \arg \min_{\tilde{\mathbf{w}}} \tilde{\mathbf{w}}^T \tilde{\Sigma} \tilde{\mathbf{w}}$ subject to $\tilde{\mathbf{w}}^T \tilde{\boldsymbol{\mu}} = \mu_*$ and $\tilde{\mathbf{w}}^T \mathbf{1} = 1$.

$\Rightarrow \tilde{\mathbf{w}}_* = (w_{r*}, \mathbf{w}_*^T)^T$, where $w_{r*} = \frac{c_{\mu\mu} - (r + \mu_*)c_{1\mu} + r\mu_*c_{11}}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$, $\mathbf{w}_* = \frac{(\mu_* - r)(\Sigma^{-1}\boldsymbol{\mu} - r\Sigma^{-1}\mathbf{1})}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$

• $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$, $c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}$, $c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$, if Σ^{-1} exists.

Example 3

Find the optimal portfolio for $\mu_* = 1.5\%$, given $r = 0.1\%$ & assets characteristics of:

$$\boldsymbol{\mu} = \begin{pmatrix} 2\% \\ 1\% \\ 1.5\% \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} 0.25\% & 0.10\% & 0.075\% \\ 0.10\% & 0.16\% & 0.12\% \\ 0.075\% & 0.12\% & 0.36\% \end{pmatrix}, \boldsymbol{\Sigma}^{-1} \approx \begin{pmatrix} 533.33 & -333.33 & 0 \\ -333.33 & 1041.67 & -277.78 \\ 0 & -277.78 & 370.37 \end{pmatrix}$$

• $\Sigma^{-1}\mathbf{1} = (200, 430.5556, 92.5926)^T$; $\Sigma^{-1}\boldsymbol{\mu} = (7.3333, -0.4167, 2.7778)^T$

• $c_{11} = 200 + 430.5556 + 92.5926 = 723.1481$

• $c_{1\mu} = 7.3333 - 0.4167 + 2.7778 = 9.6944$

• $c_{\mu\mu} = (2\%)(7.3333) + (1\%)(-0.4167) + (1.5\%)(2.7778) = 0.1842$

• $\gamma^2 = c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11} = (0.1842) - 2(0.1\%)(9.6944) + (0.1\%)^2(723.1481) = 0.1655$

• $w_{r*} = [0.1842 - (0.1\% + 1.5\%)(9.6944) + (0.1\%)(1.5\%)(723.1481)]/0.1655 = 24.11\%$

• $\mathbf{w}_* \approx \frac{1.5\% - 0.1\%}{0.1655} \left[\begin{pmatrix} 7.3333 \\ -0.4167 \\ 2.7778 \end{pmatrix} - 0.1\% \begin{pmatrix} 200 \\ 430.556 \\ 92.5926 \end{pmatrix} \right] \Rightarrow \tilde{\mathbf{w}}_* \approx (24\%, 60\%, -7\%, 23\%)^T$

Efficient Frontier, Portfolios of Risk-free & Risky Assets

MVP: $\tilde{\mathbf{w}}_* = \arg \min_{\tilde{\mathbf{w}}} \tilde{\mathbf{w}}^T \tilde{\Sigma} \tilde{\mathbf{w}}$ subject to $\tilde{\mathbf{w}}^T \tilde{\boldsymbol{\mu}} = \mu_*$ and $\tilde{\mathbf{w}}^T \mathbf{1} = 1$.

$\Rightarrow \tilde{\mathbf{w}}_* = (w_{r*}, \mathbf{w}_*^T)^T$, where $w_{r*} = \frac{c_{\mu\mu} - (r + \mu_*)c_{1\mu} + r\mu_*c_{11}}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$, $\mathbf{w}_* = \frac{(\mu_* - r)(\Sigma^{-1}\boldsymbol{\mu} - r\Sigma^{-1}\mathbf{1})}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$

• $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$, $c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}$, $c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$, if Σ^{-1} exists.

Theorem 4 (Efficient Frontier with Risk-free Assets)

For portfolios with risk-free and risky assets, let $\gamma^2 = (\boldsymbol{\mu} - r\mathbf{1})^T \Sigma^{-1} (\boldsymbol{\mu} - r\mathbf{1})$,

- The MVS is a pair of straight lines $\mu_* = \pm \gamma \sigma_* + r$
- The efficient frontier is the line $\mu_* = |\gamma| \sigma_* + r$

Proof.

Note that: $\mathbf{w}_* = (\mu_* - r)\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})\gamma^{-2}$

Recall: $\sigma_*^2 = \tilde{\mathbf{w}}_*^T \tilde{\Sigma} \tilde{\mathbf{w}}_* = \mathbf{w}_*^T \Sigma \mathbf{w}_*$

$$\begin{aligned} \Rightarrow \sigma_*^2 &= (\mu_* - r)^2 \gamma^{-4} (\boldsymbol{\mu} - r\mathbf{1})^T \Sigma^{-1} \Sigma \Sigma^{-1} (\boldsymbol{\mu} - r\mathbf{1}) \\ &= (\mu_* - r)^2 \gamma^{-2} \end{aligned}$$

$$\Rightarrow \mu_* = \pm \gamma \sigma_* + r$$

$\therefore$ MVS is 2 lines. The upper part, $\mu_* = |\gamma| \sigma_* + r$, is the efficient frontier. $\square$

Remark: The OMVP is the risk free asset $(\sigma_*, \mu_*) = (0, r)$.

Efficient Frontier Including Risk-free and Risky Assets

Efficient frontier: $\mu_* = |\gamma|\sigma_* + r$, where

$$|\gamma| = \sqrt{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}, \quad c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}, \quad c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}, \quad c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$$

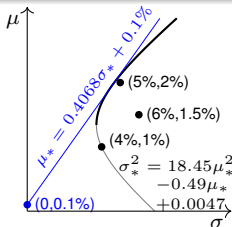
Example 4

Find the efficient frontier, given $r = 0.1\%$ & assets characteristics of:

$$\boldsymbol{\mu} = \begin{pmatrix} 2\% \\ 1\% \\ 1.5\% \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.25\% & 0.10\% & 0.075\% \\ 0.10\% & 0.16\% & 0.12\% \\ 0.075\% & 0.12\% & 0.36\% \end{pmatrix}, \quad \Sigma^{-1} \approx \begin{pmatrix} 533.33 & -333.33 & 0 \\ -333.33 & 1041.67 & -277.78 \\ 0 & -277.78 & 370.37 \end{pmatrix}$$

Recall: $c_{11} = 723.1481$, $c_{1\mu} = 9.6944$, $c_{\mu\mu} = 0.1842$, $\gamma^2 = 0.1655 \Rightarrow |\gamma| = 0.4068$

Efficient frontier: $\mu_* = 0.4068\sigma_* + 0.1\%$



Agenda

- 1 Portfolio Theory
 - Risky Assets Only
 - Risky and Risk-free Assets

- 2 Tangency Portfolio

- 3 Fund Theorems

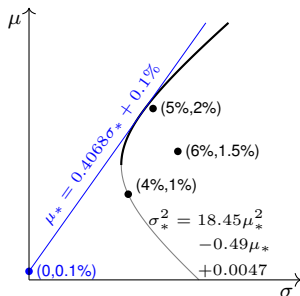
- 4 No-short-sell Constraints

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Tangency Portfolio, Introduction



- **Black:** Efficient frontier computed from (μ, Σ)
 - $\sigma_*^2 = \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu})$
- **Blue:** Efficient frontier computed from $(\tilde{\mu}, \tilde{\Sigma}) = \left(\begin{pmatrix} r \\ \mu \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \Sigma \end{pmatrix} \right)$
 - $\mu_* = |\gamma|\sigma_* + r$
- Do the lines cross, touch, or miss each other?
 - **Ans:** They touch at $w_r = 0$; o.w. contradict optimality of a frontier

Tangency Portfolio

MVP: $\tilde{\mathbf{w}}_* = \arg \min_{\tilde{\mathbf{w}}} \tilde{\mathbf{w}}^T \tilde{\Sigma} \tilde{\mathbf{w}}$ subject to $\tilde{\mathbf{w}}^T \tilde{\boldsymbol{\mu}} = \mu_*$ and $\tilde{\mathbf{w}}^T \mathbf{1} = 1$.

$\Rightarrow \tilde{\mathbf{w}}_* = (w_{r*}, \mathbf{w}_*^T)^T$, where $w_{r*} \stackrel{(a)}{=} \frac{c_{\mu\mu} - (r + \mu_*)c_{1\mu} + r\mu_*c_{11}}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$, $\mathbf{w}_* \stackrel{(b)}{=} \frac{(\mu_* - r)(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - r\boldsymbol{\Sigma}^{-1}\mathbf{1})}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$

• $c_{11} = \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}$, $c_{1\mu} = \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$, $c_{\mu\mu} = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$, if $\boldsymbol{\Sigma}^{-1}$ exists.

Theorem 5 (Tangency Portfolio)

The two efficient frontiers touch at the tangency portfolio M :

$$\mathbf{w}_M = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{(c_{1\mu} - rc_{11})}, \quad \mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}}, \quad \text{and} \quad \sigma_M = \frac{|\gamma|}{c_{1\mu} - rc_{11}}.$$

Proof.

• Tangency portfolio M corresponds to $w_r = 0$ (zero weighting for the risk-free asset):

① Put $w_r = 0$ in (a): $\mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}}$

② Sub μ_M into (b): $\mathbf{w}_M = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{(c_{1\mu} - rc_{11})}$

③ Variance: $\sigma_M^2 = \mathbf{w}_M^T \boldsymbol{\Sigma} \mathbf{w}_M = \frac{(\boldsymbol{\mu} - r\mathbf{1})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r\mathbf{1})}{(c_{1\mu} - rc_{11})^2} = \frac{\gamma^2}{(c_{1\mu} - rc_{11})^2}$,

where $\gamma^2 = (\boldsymbol{\mu} - r\mathbf{1})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r\mathbf{1}) = c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}$

$\Rightarrow \sigma_M = \frac{|\gamma|}{c_{1\mu} - rc_{11}}$

Tangency Portfolio

- (Th.2) Frontier of risky assets only: $L_2 : \sigma_*^2(\mu_*) = \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu})$
- (Th.4) Frontier of risk-free and risky assets: $L_4 : \mu_* = |\gamma|\sigma_* + r$
- $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$, $c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}$, $c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$, $\Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$,
- $\gamma^2 = c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}$, $\sigma_M = \frac{|\gamma|}{(c_{1\mu} - c_{11}r)}$, $\mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}}$

Example 5 (Gradient at M)

Show that the slopes of both frontiers L_2 and L_4 are equal at M .

Proof.

- Gradient for L_2 at M (the point (σ_M, μ_M)):

$$2\sigma_* \frac{d\sigma_*}{d\mu_*} = \Delta^{-1}(2c_{11}\mu_* - 2c_{1\mu}) \quad (\text{Differentiate both sides of } L_2)$$

$$\begin{aligned} \Rightarrow \left. \frac{d\sigma_*}{d\mu_*} \right|_{\substack{\mu_* = \mu_M \\ \sigma_* = \sigma_M}} &= \frac{c_{11}(c_{\mu\mu} - rc_{1\mu}) / (c_{1\mu} - rc_{11}) - c_{1\mu}}{\Delta|\gamma| / (c_{1\mu} - rc_{11})} \\ &= \frac{c_{11}(c_{\mu\mu} - rc_{1\mu}) - c_{1\mu}(c_{1\mu} - rc_{11})}{|\gamma|\Delta} = \frac{1}{|\gamma|} \end{aligned}$$

- Gradient for the straight line L_4 is always $\frac{d\sigma_*}{d\mu_*} = \frac{1}{|\gamma|}$

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Two-fund Theorem

- MVP: $\mathbf{w}_*(\mu_*) = \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})$
- The optimal portfolio $\mathbf{w} = \mathbf{w}_*(\mu_*)$ satisfies $\mathbf{w}^T\boldsymbol{\mu} = \mu_*$ and $\mathbf{w}^T\mathbf{1} = 1$

Theorem 6 (Two-fund Theorem)

Any *two portfolios* from the efficient frontier can reproduce the *whole frontier*.

Proof.

- Consider any 2 portfolios on the efficient frontier corresponding to $\mu_* = \mu_a$ and μ_b
$$\begin{aligned}\mathbf{w}_*(\mu_a) &= \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_a + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu}) \\ \mathbf{w}_*(\mu_b) &= \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_b + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})\end{aligned}$$
- For any $\tilde{\mu} > 0$, obtain a portfolio $\tilde{\mathbf{w}} = \alpha\mathbf{w}_*(\mu_a) + (1 - \alpha)\mathbf{w}_*(\mu_b)$ with a mean return $\tilde{\mu}$:
 - 1 Solve $\tilde{\mu} = \alpha\mu_a + (1 - \alpha)\mu_b$ for $\alpha \Rightarrow \alpha = \frac{\tilde{\mu} - \mu_b}{\mu_a - \mu_b}$
 - 2 Define $\tilde{\mathbf{w}} = \alpha\mathbf{w}_*(\mu_a) + (1 - \alpha)\mathbf{w}_*(\mu_b)$
$$\begin{aligned}&= \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})(\alpha\mu_a + (1 - \alpha)\mu_b) \\ &\quad + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu}) \\ &= \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\tilde{\mu} + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})\end{aligned}$$
- Note that $\tilde{\mathbf{w}}$ is a portfolio on the efficient frontier since $\tilde{\mathbf{w}} = \mathbf{w}_*(\tilde{\mu})$

One-fund Theorem

MVP: $\tilde{\mathbf{w}}_* = (w_{r*}, \mathbf{w}_*^T)^T$, $w_{r*} = \frac{c_{\mu\mu} - (r + \mu_*)c_{1\mu} + r\mu_*c_{11}}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$, $\mathbf{w}_* = \frac{(\mu_* - r)(\Sigma^{-1}\boldsymbol{\mu} - r\Sigma^{-1}\mathbf{1})}{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}$

• $\gamma^2 = c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}$, $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$, $c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}$, $c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$

• M is the tangency portfolio: $\mathbf{w}_M = \frac{\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{(c_{1\mu} - rc_{11})}$, $\mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}}$, $\sigma_M = \frac{|\gamma|}{c_{1\mu} - rc_{11}}$

Theorem 7 (One-fund Theorem)

The *risk-free* asset & *risky fund M* can reproduce the entire efficient frontier.

Proof.

- Consider portfolios $(1, \mathbf{0}^T)^T$ and $(0, \mathbf{w}_M^T)^T$ on the frontier corresponding to $\mu_* = r$ & μ_M ,

where $\mathbf{w}_M = \Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1}) / (c_{1\mu} - rc_{11})$

- For any $\tilde{\mu}$, obtain a portfolio $\bar{\mathbf{w}}_* = \begin{pmatrix} \alpha \\ \bar{\mathbf{w}} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} + (1 - \alpha) \begin{pmatrix} 0 \\ \mathbf{w}_M \end{pmatrix}$ with a mean return $\bar{\mu}$:

1 Solve $\bar{\mu} = \alpha r + (1 - \alpha)\mu_M$ for $\alpha \Rightarrow \alpha = \frac{c_{\mu\mu} - (r + \bar{\mu})c_{1\mu} + r\bar{\mu}c_{11}}{\gamma^2}$

2 Define $\bar{\mathbf{w}} = (1 - \alpha)\mathbf{w}_M = [(\bar{\mu} - r)\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})] / \gamma^2$

- Note that $\bar{\mathbf{w}}_* = \begin{pmatrix} \alpha \\ \bar{\mathbf{w}} \end{pmatrix}$ is a portfolio on the efficient frontier since $\alpha = w_{r*}(\bar{\mu})$ & $\bar{\mathbf{w}} = \mathbf{w}_*(\bar{\mu})$

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- 1 Portfolio Theory
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Non-negativity Constraints on Holdings

Want: $\min_{\mathbf{w} \in \mathbb{R}^n} \mathbf{w}^T \Sigma \mathbf{w}$ subject to $\mathbf{w}^T \boldsymbol{\mu} = \mu_*$ and $\mathbf{w}^T \mathbf{1} = 1$

Definition 2 (Short Selling)

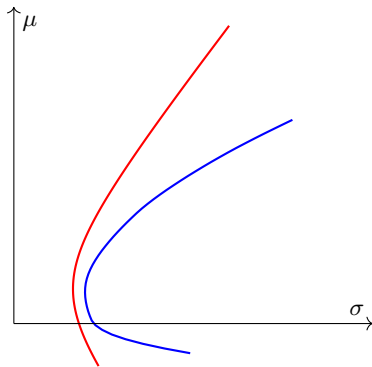
Short selling, or **shorting**, involves selling borrowed asset(s), which is/are later purchased back and returned.

- $w_i < 0$ involves short selling asset i
- Non-negativity constraints (forbidding short selling): $\mathbf{w} \geq 0$ ($w_i \geq 0, \forall i$)

Definition 3 (Minimum Variance Portfolio (MVP) w/o Shorting)

- If asset return has expectation $\boldsymbol{\mu} \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n \times n}$,
- the **Minimum Variance Portfolio (MVP) w.o. shorting** is a portfolio that
 - allocate weights $\mathbf{w}_* = (w_{*1}, \dots, w_{*n})$ on the n assets
 - satisfies $\mathbf{w}_* = \arg \min_{\mathbf{w}} \mathbf{w}^T \Sigma \mathbf{w}$ subject to $\mathbf{w}^T \boldsymbol{\mu} = \mu_*$, $\mathbf{w}^T \mathbf{1} = 1$, & $\mathbf{w} \geq 0$.
- There is no explicit solution. Quadratic programming (QP) may be used.

Remark: Non-negativity Constraints on Holdings



Non-negativity constraints affect the optimization objective function.
The two curves are the frontiers (with and without) the constraints

Non-negativity Constraints on Holdings

$$w_* = \arg \min_w w^T \Sigma w \quad \text{subject to} \quad w^T \mu = \mu_*, \quad w^T \mathbf{1} = 1, \quad \& \quad w \geq 0.$$

- `solve.QP` in R seeks $\arg \min_b (-d^T b + 0.5 b^T D b)$, sub. to $A^T b \geq b_0$, with k constraints, where $b, d \in \mathbb{R}^{n \times 1}$, $D \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times k}$, $b_0 \in \mathbb{R}^{k \times 1}$.
- Syntax: `solve.QP(Dmat, dvec, Amat, bvec, meq=0, ...)`
 - the first(rest) `meq` constraints are treated as equality(inequality) constraints.
- Implementing `solve.QP` to MVP w.o. shorting:
 - $D = \Sigma$, $d = \mathbf{0}_{n \times 1}$, $b = w$, $b_0 = (\mu_*, 1, \mathbf{0}_{1 \times n})^T$, $k = n + 2$
 - $A = (\mu_{n \times 1}, \mathbf{1}_{n \times 1}, I_{n \times n}) = \begin{pmatrix} \mu_1 & 1 & 1 & 0 & \dots & 0 \\ \mu_2 & 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_n & 1 & 0 & 0 & \dots & 1 \end{pmatrix}$, $b_0 = \begin{pmatrix} \mu_* \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
 - `meq= 2`

<https://search.r-project.org/CRAN/refmans/quadprog/html/solve.QP.html>

R Example: Non-negativity Constraints on Holdings

Example 6

Given assets characteristics: $\mu = \begin{pmatrix} 2\% \\ 1\% \\ 1.5\% \end{pmatrix}$, $\Sigma = \begin{pmatrix} 0.25\% & 0.10\% & 0.075\% \\ 0.10\% & 0.16\% & 0.12\% \\ 0.075\% & 0.12\% & 0.36\% \end{pmatrix}$,

Find the efficient frontier of risky assets with non-negativity constraint.

Notes on variable names: $M = \mu$, $C =$ Cov matrix Σ ; (Upper)lower case for (non-)scalar

Example 6 (A) Data setup

```
rm(list=ls(all=TRUE)) # clear existing objects in R
n <- 3 # Number of assets = 3
M <- c(2,1,1.5)/100 # Return vector mu
C <- matrix(c(0.25,0.1,0.075, 0.1,0.16,0.12, 0.075,0.12,0.36)/100,3,3)
C # Display Covariance matrix
      [,1] [,2] [,3]
[1,] 0.00250 0.0010 0.00075
[2,] 0.00100 0.0016 0.00120
[3,] 0.00075 0.0012 0.00360
```

R Example: Non-negativity Constraints on Holdings

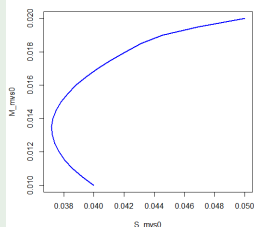
- $w_* = \arg \min_w w^T \Sigma w$ subject to $w^T \mu = \mu_*$, $w^T \mathbf{1} = 1$, & $w \geq 0$.
- R syntax: `solve.QP(Dmat, dvec, Amat, bvec, meq=0, ...)`
- Solves: $\arg \min_b (-d^T b + 0.5 b^T D b)$, sub. to $A^T b \geq b_0$
- $D = \Sigma$, $d = \mathbf{0}_{n \times 1}$, $b = w$, $b_0 = (\mu_*, 1, \mathbf{0}_{1 \times n})^T$, $A = (\mu_{n \times 1}, \mathbf{1}_{n \times 1}, I_{n \times n})$

Notes: $V_0 = \mathbf{0}$; suffix 0 for ≥ 0 constraint: $M_mvs0 = \mu_*$, $S_mvs0 = \sigma_*$, $m0 = \mu_*$, $W0 = w_*$

Example 6 (B) Non-negativity Constraints

```
library(quadprog)
ni <- 20 # Number of intervals (for plotting)
V1 <- matrix(1,n,1) # Vector of ones; n rows, 1 column
V0 <- matrix(0,n,1) # Vector of zeros; n rows, 1 column
M_mvs0 <- min(M)+(0:ni)/ni * (max(M) -min(M)) # Vector of  $\mu_*$ 's
S_mvs0 <- matrix(0,ni+1,1) # for storing  $\sigma_*$ 
Amat <- cbind(M,V1,diag(n))
for (i in c(1:(ni+1))) { # Solve MVP for each  $\mu_*$ 
  m0 <- M_mvs0[i]
  b0 <- matrix(c(m0,1,V0),n+2,1)
  p <- solve.QP(C, V0, Amat, b0, 2)
  W0 <- p$solution
  S_mvs0[i] <- sqrt(t(W0) %*% C %*% W0) # compute  $\sigma_*$ 
}
```

```
plot(S_mvs0, M_mvs0, type="l", col="blue", lwd=2)
```



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Estimation of Parameters

MVP: $\mathbf{w}_*(\mu_*) = \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})$

- $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$, $c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}$, $c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$, $\Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$
- $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$, $\Sigma = (\text{Cov}(R_i, R_j))_{1 \leq i, j \leq n} = (\sigma_{ij})_{1 \leq i, j \leq n}$

① Applying Markowitz portfolio theory requires parameters $\boldsymbol{\mu}$ & Σ

② Estimation of $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ & $\Sigma = (\sigma_{ij})_{1 \leq i, j \leq n}$:

① Collect historical data $\{R_{it}\}_{i=1, \dots, n; t=1, \dots, T}$ for T time periods

② $\hat{\mu}_i = \frac{1}{T} \sum_{t=1}^T R_{it}$

③ $\hat{\sigma}_{ij} = \frac{1}{T-1} \sum_{t=1}^T (R_{it} - \hat{\mu}_i)(R_{jt} - \hat{\mu}_j)$

③ Estimation can be highly inaccurate when n is large.

- Number of parameters in Σ is $n(n+1)/2$ (not n^2 since $\sigma_{ij} = \sigma_{ji}$)
- Recent advances for portfolio theory involve shrinkage estimation and random matrix theory

Blur of History

Question: Can estimation be improved by a higher sampling frequency?

- **Want:** $\mu_i = E(R_{it})$ where $R_{it} = \log \frac{P_{i(t+1)}}{P_{it}}$ (focus in mean for simplicity)
- **High frequency observations:** $\tilde{R}_{itq} = \log \frac{P_{i(t+q/m)}}{P_{i(t+(q-1)/m)}}$, where $q = 1, \dots, m$

$$R_{it} = \log \prod_{q=1}^m \frac{P_{i(t+q/m)}}{P_{i(t+(q-1)/m)}} = \sum_{q=1}^m \log \frac{P_{i(t+q/m)}}{P_{i(t+(q-1)/m)}} = \sum_{q=1}^m \tilde{R}_{itq}.$$

- Empirical evidence suggests that **returns from different periods are uncorrelated**: $\text{Cov}(\tilde{R}_{itq}, \tilde{R}_{isr}) = 0$ if not $t = s$ & $q = r$.
- Let $\tilde{\mu}_i = E(\tilde{R}_{itq})$, we have $\mu_i = E\left(\sum_{q=1}^m \tilde{R}_{itq}\right) = m\tilde{\mu}_i$
- Let $\tilde{\sigma}_i^2 = \text{Var}(\tilde{R}_{itq})$, we have $\sigma_i^2 = \text{Var}\left(\sum_{q=1}^m \tilde{R}_{itq}\right) = m\tilde{\sigma}_i^2$
- **Estimation using $\{\tilde{R}_{itj}\}_{i=1,\dots,n; t=1,\dots,T; j=1,\dots,m}$**
 - $\hat{\mu}_i^* = m \frac{1}{Tm} \sum_{t=1}^T \sum_{q=1}^m \tilde{R}_{itq}$

Question: Can estimation be improved by a higher sampling frequency?

- **Want:** $\mu_i = E(R_{it})$, where $R_{it} = \log \frac{P_{i(t+1)}}{P_{it}}$
- **High frequency observation:** $\tilde{R}_{itq} = \log \frac{P_{i(t+q/m)}}{P_{i(t+(q-1)/m)}}$, where $q = 1, \dots, m$
 - $\mu_i = E\left(\sum_{q=1}^m \tilde{R}_{itq}\right) = m\tilde{\mu}_i$
 - $\sigma_i^2 = m\tilde{\sigma}_i^2$
 - $\hat{\mu}_i = \frac{1}{T} \sum_{t=1}^T R_{it}$
 - $\hat{\mu}_i^* = m \frac{1}{Tm} \sum_{t=1}^T \sum_{q=1}^m \tilde{R}_{itq}$

Answer: No benefit

- $\hat{\mu}_i$:
 - $E(\hat{\mu}_i) = \frac{1}{T} \sum_{t=1}^T E(R_{it}) = \mu_i$
 - $\text{Var}(\hat{\mu}_i) = \frac{1}{T} \text{Var}(R_{it}) = \frac{\sigma_i^2}{T}$
- $\hat{\mu}_i^*$:
 - $E(\hat{\mu}_i^*) = \frac{1}{T} \sum_{t=1}^T \sum_{q=1}^m E(\tilde{R}_{itq}) = m\tilde{\mu}_i = \mu_i$
 - $\text{Var}(\hat{\mu}_i^*) = \frac{Tm}{T^2} \text{Var}(\tilde{R}_{itq}) = \frac{m\tilde{\sigma}_i^2}{T} = \frac{\sigma_i^2}{T}$

Blur of History

- $R_{it} = \log \frac{P_i(t+1)}{P_{it}}, \quad \tilde{R}_{itq} = \log \frac{P_i(t+q/m)}{P_{i(t+(q-1)/m)}}, \quad R_{it} = \sum_{q=1}^m \tilde{R}_{itq}$
- $\mu_i = E(R_{it}) = mE(\tilde{R}_{itq}) = m\tilde{\mu}_i$
- $\sigma_i^2 = \text{Var}(R_{it}) = m\text{Var}(\tilde{R}_{itq}) = m\tilde{\sigma}_i^2$

Definition 4

Consider a random variable Y with $E(Y) = \mu_Y$ and $\text{Var}(Y) = \sigma_Y^2$.
The **coefficient of variation (CV)** of Y is the **ratio of its SD to its mean**:

$$\text{CV}(Y) = \frac{\sigma_Y}{\mu_Y}$$

CV is better than σ_Y since it is **unitless** and it measures the **relative error**.

Example 7

Let R_i and $\tilde{R}_i^{(m)}$ be the yearly and $\frac{1}{m}$ -yearly returns of asset i . Find their CV.

- $\text{CV}(R_i) = \frac{\sqrt{\text{Var}(R_i)}}{E(R_i)} = \frac{\sigma_i}{\mu_i}$
- $\text{CV}(\tilde{R}_i^{(m)}) = \frac{\sqrt{\text{Var}(\tilde{R}_i^{(m)})}}{E(\tilde{R}_i^{(m)})} = \frac{\tilde{\sigma}_i}{\tilde{\mu}_i} = \frac{\sqrt{\sigma_i^2/m}}{\mu_i/m} = \sqrt{m} \frac{\sigma_i}{\mu_i} = \sqrt{m} \text{CV}(R_i)$
- The variability (in terms of CV) increases when the returns are measured in a shorter period.

Blur of History

- $R_{it} = \log \frac{P_{i(t+1)}}{P_{it}}$
- $\mu_i = E(R_{it}), \quad \sigma_i^2 = \text{Var}(R_{it})$
- $\hat{\mu}_i = \frac{1}{T} \sum_{t=1}^T R_{it}$
 - $E(\hat{\mu}_i) = \frac{1}{T} \sum_{t=1}^T E(R_{it}) = \mu_i$
 - $\text{Var}(\hat{\mu}_i) = \frac{1}{T} \text{Var}(R_{it}) = \frac{\sigma_i^2}{T}$
- **Coefficient of variation:** $\text{CV}(Y) = \frac{\sigma_Y}{\mu_Y}$

Example 8

If $\mu_i = 0.12$ and $\sigma_i = 0.15$. Find the number of years T such that $\text{CV}(\hat{\mu}_i) \leq 0.1$.

- $\text{CV}(\hat{\mu}_i) = \frac{\sqrt{\text{Var}(\hat{\mu}_i)}}{E(\hat{\mu}_i)} = \frac{\sigma_i/\sqrt{T}}{\mu_i} = \frac{0.15}{0.12\sqrt{T}} = \frac{1.25}{\sqrt{T}}$
- $\frac{1.25}{\sqrt{T}} \leq 0.1 \quad \Rightarrow \quad T \geq 156.25$
- It is difficult to measure **mean return μ** with a high accuracy using historical data

Agenda

- 1 Portfolio Theory
 - Risky Assets Only
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- 6 Real Data Application

- 7 References

R Example: Building the efficient frontier

Example 9

- 1 Find the returns of 20 selected HK shares
- 2 Build MVS for a) risky assets, b) with risk-free assets
- 3 Find the tangency portfolio

Example 9 (A) Download data)

```
library("tseries")
library("quantmod") # bug fix for tseries

# A.1) Initialize download parameters
IDs <- matrix(c("1038", "1093", "2688", "0002", "0003", "1398", "1109",
"2628", "0017", "0386", "1044", "0669", "0012", "2319", "0267", "0883",
"0992", "2331", "0101", "0027"))
n <- length(IDs) # Number of Stocks is 20
d0="2017-07-01"; d1="2022-07-01"; Q="AdjClose" # Parameters
Data <- matrix(0,5*12+1,n) # Get (5+1/12)Yr prices to store 5Yr returns

# A.2) Download
for (i in c(1:n)) {
  Sec <- paste0(IDs[i], ".HK") # paste0 gives no separation
  Data[,i] <- get.hist.quote(Sec,d0,d1,Q,compression="m") # m:monthly
}
```

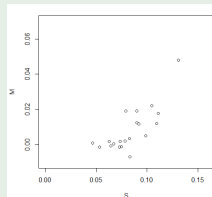
R Example: Building the efficient frontier

Notes on variable names: $R = R = (r_{ti}) = R_i$ in period t , $M = \mu$, $S = \sigma$, $C =$ Cov matrix Σ

Example 9 (B) Data preparation

```
# B.1) Calculate returns from closing prices
nP <- dim(Data)[1]      # Number of Prices (rows) is 61
R <- diff(as.matrix(Data))/as.matrix(Data[-nP,]) # Compute returns
R      # Check
#           [,1]          [,2]          [,3] ...
# [1,] -0.026767044  0.003283976 -0.050895349 ...
# [2,] -0.043092330  0.070376403  0.124131112 ...
# [3,]  0.010416865  0.036697281  0.010600703 ...

# B.2) Calculate market data statistics:
nR <- dim(R)[1]        # Number of Returns (rows) is 60
M <- apply(R,2,mean)    # Average by column
C <- cov(R)             # Cov matrix of securities
S <- sqrt(diag(C))      # Vector of sigmas
slim <- c(0,0.16); mlim <- c(-0.01,0.07) #Plot's range
plot(S, M, xlim=slim, ylim=mlim) # Plot 20 stocks ( $\sigma, \mu$ )
```



R Example: Building the efficient frontier

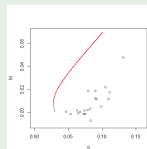
- **MVS:** $\sigma_*^2(\mu_*) = \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu}); \quad \mathbf{w}_{\min\text{var}} = \Sigma^{-1}\mathbf{1}/c_{11}$
- $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}, \quad c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}, \quad c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}, \quad \Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$

$\mathbf{M} = \boldsymbol{\mu}, \quad \mathbf{S} = \boldsymbol{\sigma}, \quad \mathbf{C} = \Sigma, \quad \mathbf{V1} = \mathbf{1}, \quad \mathbf{d} = \Delta, \quad \mathbf{m} = \mu_{\min\text{var}} = c_{1\mu}/c_{11}, \quad \mathbf{M_mvs} = \boldsymbol{\mu}_*, \quad \mathbf{S_mvs} = \boldsymbol{\sigma}_*$

Example 9 (C) Build the theoretical efficient frontier

```
# C.1) Calculates the parameters and constants of the frontier
CInv <- solve(C)      # Inverse of cov matrix
V1 <- matrix(1,n,1)   # A n x 1 vector of 1's
c11 <- as.numeric(t(V1) %*% CInv %*% V1) # make it a scalar
clm <- as.numeric(t(V1) %*% CInv %*% M)
cmm <- as.numeric(t(M) %*% CInv %*% M)
d <- c11 * cmm - clm^2
m <- clm/c11          #  $\mu_{\min\text{var}}$  from overall min variance portfolio
c(c11, clm, cmm, d, m*100)
# [1] 1293.4981737   10.5175150   0.4894047  522.4260150   0.8131084

# C.2) Set mus and find MVPs on the MVS
M_mvs <- seq(0.001,0.07,length.out=50) # Set vector of  $\mu_*$ 
S_mvs <- sqrt((c11 * M_mvs^2 - 2 * clm * M_mvs + cmm)/d)
M_ef <- M_mvs[M_mvs>=m]; S_ef <- S_mvs[M_mvs>=m] # Frontier
lines(S_mvs, M_mvs) # Draw min variance set
lines(S_ef, M_ef, col="red",lwd=2) # Draw efficient frontier
```



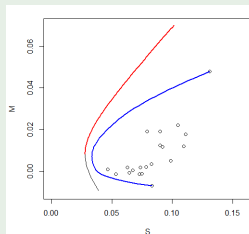
Non-negativity Constraints on Holdings

- $\mathbf{w}_* = \arg \min_{\mathbf{w}} \mathbf{w}^T \Sigma \mathbf{w}$ subject to $\mathbf{w}^T \boldsymbol{\mu} = \mu_*$, $\mathbf{w}^T \mathbf{1} = 1$, & $\mathbf{w} \geq 0$.
- R syntax: `solve.QP(Dmat, dvec, Amat, bvec, meq=0, ...)`
- Solves: $\arg \min_{\mathbf{b}} (-\mathbf{d}^T \mathbf{b} + 0.5 \mathbf{b}^T \mathbf{D} \mathbf{b})$, sub. to $\mathbf{A}^T \mathbf{b} \geq \mathbf{b}_0$
- $\mathbf{d} = \mathbf{0}_{n \times 1}$, $\mathbf{b} = \mathbf{w}$, $\mathbf{D} = \Sigma$, $\mathbf{b}_0 = (\mu_*, 1, \mathbf{0}_{1 \times n})^T$, $\mathbf{A} = (\boldsymbol{\mu}_{n \times 1}, \mathbf{1}_{n \times 1}, \mathbf{I}_{n \times n})$

Notes: $V0=0$; suffix 0 for ≥ 0 constraint: $M_mvs0 = \mu_*$, $S_mvs0 = \sigma_*$, $m0 = \mu_*$, $W0 = \mathbf{w}_*$

Example 9 (D) Vs MVS with Non-negativity Constraints

```
library(quadprog)
ni <- 50 # Number of  $\mu$  grid-points on vertical axis
V0 <- matrix(0,n,1) #  $n$  by 1 column vector of zeros
M_mvs0 <- seq(min(M)+0.001,max(M)-0.001,length.out=ni) # Vector of  $\mu_*$ 
S_mvs0 <- matrix(0,ni,1) # For storing  $\sigma_*$ 
Amat <- cbind(M,V1,diag(n))
for (i in 1:ni) {
  m <- M_mvs0[i]
  b0 <- matrix(c(m,1,V0),n+2,1)
  p <- solve.QP(C, V0, Amat, b0, 2)
  W <- p$solution # optimal weight
  S_mvs0[i] <- sqrt(t(W) %*% C %*% W) #  $\sigma_*$ 
}
lines(S_mvs0, M_mvs0, col="blue", lwd=2) # Plot MVS
```



Efficient Frontier Including Risk-free and Risky Assets

Efficient frontier: $\mu_* = |\gamma|\sigma_* + r$, where

$$|\gamma| = \sqrt{c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11}}, \quad c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}, \quad c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}, \quad c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$$

Notes on variable naming: $g = \gamma$, $mp = \mu_*$, $sp = \sigma_*$

Example 9 (E) Draw frontier inc. risk-free & risky assets)

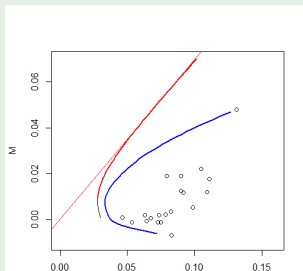
```
# E.1) Find gamma
```

```
r <- 0.02046/100 # risk-free rate
```

```
g <- sqrt(cmm - 2 * r * c1m + r^2 * c11) # gamma
```

```
# E.3) Plot efficient frontier
```

```
abline(r,g, col="red") # a line with intercept r and slope g
```



R Example: Tangency Portfolio

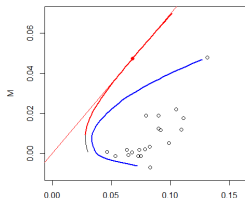
$$\text{Tangency portfolio } M: \quad \mathbf{w}_M = \frac{\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{(c_1\boldsymbol{\mu} - rc_{11})}, \quad \mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_1\boldsymbol{\mu} - rc_{11}}, \quad \text{and} \quad \sigma_M = \frac{|\gamma|}{c_1\boldsymbol{\mu} - rc_{11}}.$$

Notes on variable naming: $\mathbf{w}_M = \mathbf{w}_m$, $\mu_M = \mu_m$, $\sigma_M = \sigma_m$

Example 9 (F) Tangency portfolio

```
# F.1) Find sigma and mu of tangency portfolio M
sM <- g / (c1m - r * c11)          #  $\sigma_M$ 
mM <- (cmm - r * c1m) / (c1m - r * c11)  #  $\mu_M$ 
WM <- CInv %*% (M - r * V1) / (c1m - r * c11)  #  $\mathbf{w}_M$ 
points(sM, mM, pch=16, col="red") # Tangency portfolio point
c(sM, mM, sum(WM)) # Checking
# [1] 0.06793523 0.04752358 1.00000000
```

- The tangency portfolio achieves $(\sigma_M, \mu_M) = (6.79\%, 4.75\%)$



Summary

- When **only risky assets** are investable,
 - (MVP Solution) If Σ^{-1} exists, a target return μ_* can be obtained by the portfolio $\mathbf{w}_*(\mu_*) = \Delta^{-1}(c_{11}\Sigma^{-1}\boldsymbol{\mu} - c_{1\mu}\Sigma^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\Sigma^{-1}\mathbf{1} - c_{1\mu}\Sigma^{-1}\boldsymbol{\mu})$, where $c_{11} = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$; $c_{1\mu} = \mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}^T\Sigma^{-1}\mathbf{1}$; $c_{\mu\mu} = \boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}$; $\Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$
 - (Efficient frontier) is hyperbola: $\sigma_*^2 = \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu})$, with $\mu_* \geq \mu_{\min\text{var}}$
 - (Two-fund theorem) Any two portfolios from the frontier can reproduce the whole frontier.
- When **both risky and risk-free assets** are investable
 - (MVP Solution) If Σ^{-1} exists, a target return μ_* is obtained by $\tilde{\mathbf{w}}_* := (w_{r*}, \mathbf{w}_*^T)^T$
 $w_{r*} = (\boldsymbol{\mu} - \mu_*\mathbf{1})^T\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})\gamma^{-2}$ and $\mathbf{w}_* = (\mu_* - r)\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})\gamma^{-2}$, where $\gamma^2 = (\boldsymbol{\mu} - r\mathbf{1})^T\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})$
 - (Efficient frontier) is the line $\mu_* = |\gamma|\sigma_* + r$; MVS is $\mu_* = \pm\gamma\sigma_* + r$; $\sigma_{\min} = 0$
 - (Tangency portfolio) M has: $\mathbf{w}_M = \frac{\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{(c_{1\mu} - rc_{11})}$, $\mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}}$, $\sigma_M = \frac{|\gamma|}{c_{1\mu} - rc_{11}}$
 - (One-fund theorem) The risk-free asset & risky fund M can reproduce the whole frontier.
- **Restriction on short-selling:** $w_i \geq 0, \forall i$,
 - MVP: $\mathbf{w}_* = \arg \min_{\mathbf{w}} \mathbf{w}^T\Sigma\mathbf{w}$ subject to $\mathbf{w}^T\boldsymbol{\mu} = \mu_*$, $\mathbf{w}^T\mathbf{1} = 1$, & $\mathbf{w} \geq 0$
 - No explicit solution $\Rightarrow$ Quadratic programming (QP) $\Rightarrow$ efficient frontier is different.

Agenda

- 1 Portfolio Theory
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- 5 Estimation of Parameters

- 6 Real Data Application

- 7 References

References

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