

Arbitrage Pricing Theory (APT)

Chapter 3: Arbitrage Pricing Theory

CAPM: $\mu_a = \beta_a(\mu_M - r) + r$, where $\beta_a = \sigma_{aM} / \sigma_M^2$, $\mu_a = E(R_a)$, $\mu_M = E(R_M)$

- CAPM is an **equilibrium** model based on a **single factor** $(\mu_M - r)$.
- Limitations of CAPM include:
 - ❶ Lack of explanatory power using a single factor.
 - ❷ Lack of economic content (ignored GDP, inflation, etc...)
 - ❸ Relies on mean-variance theory in terms of tangency portfolio M .
 - ❹ Considers only the equilibrium condition of the market
 - ❺ Identification of the market portfolio is difficult
- **Arbitrage Pricing Theory** solves ❶ to ❺ based on **multifactor models**.

Agenda

- 1 Factor Models
 - Multifactor Model
 - Choice of Factors
 - Fama-French Factor Models
- 2 Arbitrage Pricing Theory (APT)
- 3 Estimation
 - Time Series Regression
 - Cross-Sectional Regression
 - R Application
- 4 Summary
- 5 References

Multifactor Model

Definition 1 (Multifactor Model)

A K -factor **multifactor model** for asset $i \in \{1, \dots, n\}$ is defined as

$$\begin{aligned} R_{it} &= a_i + b_{i1}F_{t1} + \dots + b_{iK}F_{tK} + e_{it} \\ &= a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it}, \end{aligned}$$

where

- R_{it} is the return of asset i in period t
- $\mathbf{F}_t = (F_{t1}, \dots, F_{tK})^T \in \mathbb{R}^K$ are **factors** at time t
 - $\boldsymbol{\mu}_F := \mathbb{E}(\mathbf{F}_t)$
 - $\boldsymbol{\Omega}_F := \text{Cov}(\mathbf{F}_t)$
- $\{e_{it}\}$ are errors satisfying
 - $\mathbb{E}(e_{it}) = 0, \quad \text{Var}(e_{it}) = \sigma_{ei}^2, \quad \mathbb{E}(e_{it}e_{jt}) = 0, \forall i \neq j$
 - $\text{Cov}(e_{it}, \mathbf{F}_t) = \mathbf{0}, \forall i \text{ and } t$
- a_i is an intercept term for asset i
- $\mathbf{b}_i = (b_{i1}, \dots, b_{iK})^T \in \mathbb{R}^K$ are **factor loadings/sensitivities** for asset i

Multifactor Model

- Multifactor model** for asset $i \in \{1, \dots, n\}$: $R_{it} = a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it}$
- $\mathbf{F}_t = (F_{t1}, \dots, F_{tK})^T \in \mathbb{R}^K$, $\boldsymbol{\mu}_F = \mathbb{E}(\mathbf{F}_t)$, $\boldsymbol{\Omega}_F = \text{Cov}(\mathbf{F}_t)$
 - $\mathbb{E}(e_{it}) = 0$, $\text{Var}(e_{it}) = \sigma_{ei}^2$, $\mathbb{E}(e_{it}e_{jt}) = 0$, $\text{Cov}(e_{it}, \mathbf{F}_t) = \mathbf{0}$

Theorem 1

For a portfolio $\mathbf{w} = (w_1, \dots, w_n)^T \in \mathbb{R}^n$ on the n -assets, its return is

$$R_{pt} = \mathbf{w}^T \mathbf{a} + \mathbf{w}^T \mathbf{B} \mathbf{F}_t + \mathbf{w}^T \mathbf{e}_t,$$

where

- $\mathbf{a} = (a_1, \dots, a_n)^T$, $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)^T \in \mathbb{R}^{n \times K}$, $\mathbf{e}_t = (e_{1t}, \dots, e_{nt})^T \in \mathbb{R}^n$

Moreover, the mean and variance of the portfolio return is

- $\mu_p = \mathbf{w}^T \mathbf{a} + \mathbf{w}^T \mathbf{B} \boldsymbol{\mu}_F$
- $\sigma_p^2 = \underbrace{\mathbf{w}^T \mathbf{B} \boldsymbol{\Omega}_F \mathbf{B}^T \mathbf{w}}_{\text{factor risk}} + \underbrace{\mathbf{w}^T \boldsymbol{\Psi}_e \mathbf{w}}_{\text{non-systematic risk}}$, where $\boldsymbol{\Psi}_e = \text{Diag}(\sigma_{e1}^2, \dots, \sigma_{en}^2)$

- **Proof:** Note that $\mathbf{R}_t = (R_{1t}, \dots, R_{nt})^T = \mathbf{a} + \mathbf{B} \mathbf{F}_t + \mathbf{e}_t$ and $R_{pt} = \mathbf{w}^T \mathbf{R}_t$

Multifactor Model

Multifactor model: $\mathbf{R}_t = \mathbf{a} + \mathbf{B}\mathbf{F}_t + \mathbf{e}_t$, $\mathbf{F}_t \in \mathbb{R}^K$, $\boldsymbol{\mu}_F := \mathbb{E}(\mathbf{F}_t)$, $\boldsymbol{\Omega}_F := \text{Cov}(\mathbf{F}_t)$
For a portfolio ($n \geq 1$): $R_{pt} = \mathbf{w}^T \mathbf{a} + \mathbf{w}^T \mathbf{B} \mathbf{F}_t + \mathbf{w}^T \mathbf{e}_t$, where $\mathbf{w} \in \mathbb{R}^n$, $\mathbf{B} \in \mathbb{R}^{n \times K}$
 • $\mu_p = \mathbf{w}^T \mathbf{a} + \mathbf{w}^T \mathbf{B} \boldsymbol{\mu}_F$, $\sigma_p^2 = \mathbf{w}^T \mathbf{B} \boldsymbol{\Omega}_F \mathbf{B}^T \mathbf{w} + \mathbf{w}^T \boldsymbol{\Psi}_e \mathbf{w}$, $\boldsymbol{\Psi}_e = \text{Diag}(\sigma_{e1}^2, \dots, \sigma_{en}^2)$

Example 1

Consider # assets $n = 2$, # factors $K = 3$, portfolio weight $\mathbf{w} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$, and

$$\mathbf{a} = \begin{pmatrix} 1\% \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0.25 & 0.5 \\ 0.8 & 0.5 & 0.4 \end{pmatrix}, \boldsymbol{\mu}_F = \begin{pmatrix} 8\% \\ 0\% \\ 2\% \end{pmatrix}, \boldsymbol{\Omega}_F = \begin{pmatrix} 3\% & 0.4\% & -0.8\% \\ 0.4\% & 1\% & -0.1\% \\ -0.8\% & -0.1\% & 2\% \end{pmatrix}, \boldsymbol{\Psi}_e = \begin{pmatrix} 24\% & 0 \\ 0 & 16\% \end{pmatrix}$$

- Expected portfolio return:

$$\mu_p = \mathbf{w}^T \mathbf{a} + \mathbf{w}^T \mathbf{B} \boldsymbol{\mu}_F = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \left[\begin{pmatrix} 1\% \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0.25 & 0.5 \\ 0.8 & 0.5 & 0.4 \end{pmatrix} \begin{pmatrix} 8\% \\ 0\% \\ 2\% \end{pmatrix} \right] = 8.6\%$$

- Variance of portfolio return:

$$\begin{aligned} \sigma_p^2 &= \mathbf{w}^T \mathbf{B} \boldsymbol{\Omega}_F \mathbf{B}^T \mathbf{w} + \mathbf{w}^T \boldsymbol{\Psi}_e \mathbf{w} = \mathbf{w}^T (\mathbf{B} \boldsymbol{\Omega}_F \mathbf{B}^T + \boldsymbol{\Psi}_e) \mathbf{w} \\ &= \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \left[\begin{pmatrix} 1 & 0.25 & 0.5 \\ 0.8 & 0.5 & 0.4 \end{pmatrix} \begin{pmatrix} 3\% & 0.4\% & -0.8\% \\ 0.4\% & 1\% & -0.1\% \\ -0.8\% & -0.1\% & 2\% \end{pmatrix} \begin{pmatrix} 1 & 0.8 \\ 0.25 & 0.5 \\ 0.5 & 0.4 \end{pmatrix} + \begin{pmatrix} 24\% & 0 \\ 0 & 16\% \end{pmatrix} \right] \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \\ &\approx 12.6\% \end{aligned}$$

- Factor Risk = $\mathbf{w}^T \mathbf{B} \boldsymbol{\Omega}_F \mathbf{B}^T \mathbf{w} = 2.6\%$; Non-systematic risk = $\mathbf{w}^T \boldsymbol{\Psi}_e \mathbf{w} = 10\%$

Choice of Factors

Three main types of factors $\{F_{it}\}_{t=1,\dots}$:

1 Macroeconomic factors

- Observable economic variables
- Examples:
 - GDP, yield spread, consumer price index (inflation), etc...
 - "Surprise" or "Residual" variable:

$$X_t - E(X_t | \mathcal{F}_{t-1}), \text{ or } X_t - r_t$$

where $\mathcal{F}_{t-1}$ is the information up till time $t - 1$; r_t is risk-free rate

2 Fundamental factors

- Observable company features
- Examples:
 - dividend yield, payout ratio, price to book (P/B) ratio, earning to price ratio, etc...

3 Statistical factors

- Unobservable factors estimated from data (e.g. principle component analysis (PCA))
- Difficulty to be interpreted or used for forecasting

Fama-French Three-Factor Model

Multifactor model for asset $i \in \{1, \dots, n\}$: $R_{it} = a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it}$

Definition 2 (Fama-French Three-Factor Model)

$$R_{it} - r_t = a_i + b_i(R_{Mt} - r_t) + s_i SMB_t + h_i HML_t + e_{it}$$

- r_t = risk-free rate
- R_{it} = Return of asset i
- $R_{Mt} = \sum_{i=1}^n w_i R_{it} / \sum_{i=1}^n w_i$ = Return of market portfolio (w_i = market capitalization)
- **Small Minus Big:** $SMB_t = R_{\text{Small},t} - R_{\text{Big},t}$
 - $\tilde{w}$ = median($w_1, \dots, w_n$) is median market capitalization of the n -assets
 - $R_{\text{Small},t} = \left(\sum_{i=1}^n w_i 1_{\{w_i \leq \tilde{w}\}} \right)^{-1} \sum_{i=1}^n w_i 1_{\{w_i \leq \tilde{w}\}} R_{it}$
 - $R_{\text{Big},t} = \left(\sum_{i=1}^n w_i 1_{\{w_i > \tilde{w}\}} \right)^{-1} \sum_{i=1}^n w_i 1_{\{w_i > \tilde{w}\}} R_{it}$
- **High Minus Low:** $HML_t = R_{\text{High},t} - R_{\text{Low},t}$
 - $q_\alpha = \alpha$ -quantile of book-to-market ratios $\{z_1, \dots, z_n\}$ of the n -assets
 - $R_{\text{High},t} = \left(\sum_{i=1}^n w_i 1_{\{z_i > q_{0.7}\}} \right)^{-1} \sum_{i=1}^n w_i 1_{\{z_i > q_{0.7}\}} R_{it}$
 - $R_{\text{Low},t} = \left(\sum_{i=1}^n w_i 1_{\{z_i \leq q_{0.3}\}} \right)^{-1} \sum_{i=1}^n w_i 1_{\{z_i \leq q_{0.3}\}} R_{it}$
- a_i, b_i, s_i, h_i are factor sensitivities

Fama-French 3-factor model: $R_{it} - r_t = a_i + b_i(R_{Mt} - r_t) + s_iSMB_t + h_iHML_t + e_{it}$

- r_t = risk-free rate; R_{it} = Return of asset i
- $R_{Mt} = \sum_{i=1}^n w_i R_{it}$ = Return of market portfolio (w_i = market capitalization)
- **Small Minus Big:** $SMB_t = R_{Small,t} - R_{Big,t}$ (S/B market capitalization)
- **High Minus Low:** $HML_t = R_{High,t} - R_{Low,t}$ (H/L book-to-market ratio)

Interpretation of factors:

- 1 R_{Mt} : Inherited from CAPM
- 2 **Small Minus Big** SMB_t
 - Market capitalization is the total market value of a company's stocks
 - Also known as **small firm effect** or **size effect**
 - Measures how small cap companies outperform large cap companies
 - **Empirical findings:** s_i higher if portfolio i has more small-cap companies
- 3 **High Minus Low** HML_t
 - Book-to-market (B/M) ratio = $\frac{\text{accounting book value (equity)}}{\text{market capitalization}}$
 - High B/M $\Leftrightarrow$ Company Equity > Market Price $\Leftrightarrow$ Stock is cheap
 - Also known as **value** or **value versus growth** factor
 - Stocks with **high/normal/low** B/M ratio: **value/neutral/growth** stocks
 - Measures how high B/M companies outperform low B/M companies
 - **Empirical findings:** h_i higher if portfolio i has more high B/M companies

Fama-French Five-Factor Model

Fama-French 3-factor model: $R_{it} - r_t = a_i + b_i(R_{Mt} - r_t) + s_iSMB_t + h_iHML_t + e_{it}$

Definition 3 (Fama-French Five-Factor Model)

$$R_{it} - r_t = a_i + b_i(R_{Mt} - r_t) + s_iSMB_t + h_iHML_t + r_iRMW_t + c_iCMA_t + e_{it}$$

- r_t = risk-free rate; R_{it} = Return of asset i ; R_{Mt} = Market return

- **Small Minus Big:** $SMB_t = R_{Small,t} - R_{Big,t}$

- **High Minus Low:** $HML_t = R_{High,t} - R_{Low,t}$

- **Robust Minus Weak:** $RMW_t = R_{Robust,t} - R_{Weak,t}$

- $q_\alpha = \alpha$ -quantile of **operating profitability** $\{z_1, \dots, z_n\}$ of the n -assets

- $R_{Robust,t} = \left(\sum_{i=1}^n w_i 1_{\{z_i > q_{0.7}\}} \right)^{-1} \sum_{i=1}^n w_i 1_{\{z_i > q_{0.7}\}} R_{it}$

- $R_{Weak,t} = \left(\sum_{i=1}^n w_i 1_{\{z_i \leq q_{0.3}\}} \right)^{-1} \sum_{i=1}^n w_i 1_{\{z_i \leq q_{0.3}\}} R_{it}$

- **Conservative Minus Aggressive:** $HML_t = R_{Conser,t} - R_{Aggre,t}$

- $q_\alpha = \alpha$ -quantile of **investment** $\{z_1, \dots, z_n\}$ of the n -assets

- $R_{Aggre,t} = \left(\sum_{i=1}^n w_i 1_{\{z_i > q_{0.7}\}} \right)^{-1} \sum_{i=1}^n w_i 1_{\{z_i > q_{0.7}\}} R_{it}$

- $R_{Conser,t} = \left(\sum_{i=1}^n w_i 1_{\{z_i \leq q_{0.3}\}} \right)^{-1} \sum_{i=1}^n w_i 1_{\{z_i \leq q_{0.3}\}} R_{it}$

- $a_i, b_i, s_i, h_i, r_i, c_i$ are **factor sensitivities**

Fama-French Five-Factor Model

Fama-French 5: $R_{it} - r_t = a_i + b_i(R_{Mt} - r_t) + s_i SMB_t + h_i HML_t + r_i RMW_t + c_i CMA_t + e_{it}$

- r_t = risk-free rate; R_{it} = Return of asset i ; R_{Mt} = Market return
- **Small Minus Big:** SMB_t ; **High Minus Low:** HML_t
- **Robust Minus Weak:** $RMW_t = R_{Robust,t} - R_{Weak,t}$ (R/W Operating Profitability)
- **Conservative Minus Aggressive:** $HML_t = R_{Conser,t} - R_{Aggre,t}$ (C/A Investments)

Interpretation of factors:

- 1 R_{Mt} ; SMB_t , HML_t : Inherited from Fama-French 3-factor model
- 2 **Robust Minus Weak** RMW_t
 - Operating Profitability (OP) = $\frac{\text{net revenues}}{\text{accounting book value (equity)}}$
 - Measures how robust ($\uparrow$ OP) firms outperform weak ($\downarrow$ OP) ones
 - **Empirical findings:** r_i higher if portfolio i has more robust companies
- 3 **Conservative Minus Aggressive** CMA_t
 - Investments (Inv) = % changes in total assets in the previous period
 - due to **new fundings** & **profit not paid out as dividends**
 - Measures how conservative ($\downarrow$ Inv) firms outperform aggressive ($\uparrow$ Inv) ones
 - **Empirical findings:** c_i higher if portfolio i has more low Inv companies

Empirical Analysis of Asset Prices



In an efficient market at any point in time the actual price of a security will be a good estimate of its intrinsic value.

— Eugene Fama —

AZ QUOTES

Eugene F. Fama (1939~): Nobel Economics 2013

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Arbitrage Pricing Theory (APT)

CAPM: $\mu_a = \beta_a(\mu_M - r) + r$, where $\beta_a = \sigma_{aM}/\sigma_M^2$, $\mu_a = E(R_a)$, $\mu_M = E(R_M)$

- The derivation of CAPM relies on mean-variance theory in terms of the tangency portfolio M
- APT derives similar expected return relations based solely on “no arbitrage” arguments
- APT Assumptions
 - 1 Large number of investible assets ($n \rightarrow \infty$)
 - 2 No arbitrage opportunities in an efficient market

Theorem 2 (Arbitrage Pricing Theory)

Consider n assets under the K -factor model $R_i = a_i + \mathbf{b}_i^T \mathbf{F} + e_i; i = 1, \dots, n$. There exist constants $\lambda_0, \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_K)^T$ (factor risk premiums) such that

$$\mu_i = \lambda_0 + \mathbf{b}_i^T \boldsymbol{\lambda}.$$

Note: Merely taking expectation $\mu_i = E(R_i) = a_i + \mathbf{b}_i^T E(\mathbf{F})$ cannot guarantee $a_i = \lambda_0$ for all i .

Arbitrage Pricing Theory (APT)

APT: $\mu_i = \lambda_0 + \mathbf{b}_i^T \boldsymbol{\lambda}$ under the K -factor model $R_i = a_i + \mathbf{b}_i^T \mathbf{F} + e_i; i = 1, \dots, n$

- ① Large number of investible assets ($n \rightarrow \infty$)
- ② No arbitrage opportunities in an efficient market

Multifactor model: $\mathbf{R}_t = \mathbf{a} + \mathbf{B}\mathbf{F}_t + \mathbf{e}_t \in \mathbb{R}^n$, $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)^T \in \mathbb{R}^{n \times K}$

Proof.

- WLOG, let the K columns of $\mathbf{B}$ be linearly independent (each factor is unique)
- ⇒ Exist orthogonal $\mathbf{w}_1, \dots, \mathbf{w}_{n-K} \in \mathbb{R}^n$ s.t. $\mathbf{w}_j^T \mathbf{1} = 1$ & $\mathbf{w}_j^T \mathbf{B} = \mathbf{0}$ for all j
- By ①, we have $\mathbf{w}^T \mathbf{e}_t = 0$ for any well-diversified portfolio $\mathbf{w}$.
- By ②, maturity value $\mathbf{w}^T \mathbf{R}_t = r$ if it's nonrandom & initial value $\mathbf{w}^T \mathbf{1} = 1$.
- $\mathbf{w}_j^T \mathbf{R}_t = \mathbf{w}_j^T (\mathbf{a} + \mathbf{B}\mathbf{F}_t + \mathbf{e}_t) = \mathbf{w}_j^T \mathbf{a} = r \implies (\mathbf{a} - r\mathbf{1}) \perp \mathbf{w}_1, \dots, \mathbf{w}_{n-K}$
- Note that $\mathbf{B}, \mathbf{w}_1, \dots, \mathbf{w}_{n-K}$ span $\mathbb{R}^n$, so $\mathbf{a} - r\mathbf{1} \in \mathbb{R}^n$ is spanned by $\mathbf{B}$
 - ⇒ $\mathbf{a} - r\mathbf{1} = \mathbf{B}\tilde{\boldsymbol{\lambda}}$
 - ⇒ $\boldsymbol{\mu} = E(\mathbf{R}_t) = \mathbf{a} + \mathbf{B}E(\mathbf{F}_t) = r\mathbf{1} + \mathbf{B}(\tilde{\boldsymbol{\lambda}} + \boldsymbol{\mu}_F) =: \lambda_0 \mathbf{1} + \mathbf{B}\boldsymbol{\lambda}$



APT & Risk Premium

CAPM: $\mu_a = r + \beta_a(\mu_M - r)$, where $\beta_a = \sigma_{aM}/\sigma_M^2$, $\mu_a = E(R_a)$, $\mu_M = E(R_M)$

APT: $\mu_i = \lambda_0 + \mathbf{b}_i^T \boldsymbol{\lambda}$ under the K -factor model $R_i = a_i + \mathbf{b}_i^T \mathbf{F} + e_i$; $i = 1, \dots, n$

- λ_0 and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_K)$ are **factor risk premiums**.

Interpretations and Implications of APT

• Factor Risk Premium:

- Recall $\lambda_0 = r$ from the proof of APT
- Write $\boldsymbol{\mu} = r\mathbf{1} + \mathbf{B}\boldsymbol{\lambda}$, where $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)^T \in \mathbb{R}^{n \times K}$.
- If columns of $\mathbf{B}$ are linearly independent, then we can find (Gram-Schmidt) a $\mathbf{w}_i$ s.t.

$$\mathbf{w}_i^T \mathbf{B} = (0, \dots, 1, \dots, 0) \text{ (} i\text{-th entry is 1)}$$

① $\mathbf{w}_i^T \mathbf{R} = \mathbf{w}_i^T (\mathbf{a} + \mathbf{B}\mathbf{F} + \mathbf{e}) = \mathbf{w}_i^T \mathbf{a} + F_i + \mathbf{w}_i^T \mathbf{e}$

⇒ $\mathbf{w}_i^T$ is a portfolio representing the i -th factor (Arrow-Debreu security)

② $\mathbf{w}_i^T (\boldsymbol{\mu} - r\mathbf{1}) = \mathbf{w}_i^T \mathbf{B}\boldsymbol{\lambda} = (0, \dots, 1, \dots, 0)\boldsymbol{\lambda} = \lambda_i$

⇒ λ_i is the expected excess return of the portfolio $\mathbf{w}_i$

③ In CAPM, the excess return $\mu_M - r$ is called the **market risk premium**

- Combining ① ② ③, λ_i is the **factor risk premium** for the i -th factor
- Similar to CAPM, APT relates an asset's return to excess returns of some factors

Beyond The Limitations of CAPM

CAPM: $\mu_i = r + \beta_i(\mu_M - r)$, where $\beta_i = \sigma_{iM}/\sigma_M^2$, $\mu_i = E(R_i)$, $\mu_M = E(R_M)$

Multifactor model: $R_{it} = a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it}$

APT: $\mu_i = \lambda_0 + \mathbf{b}_i^T \boldsymbol{\lambda}$

Overcoming the limitations of CAPM

- ① Lack of explanatory power using a single factor.
 - Multifactor model allows multiple factors
- ② Lack of economic content (ignored GDP, inflation, etc...)
 - Macroeconomic and fundamental factors can be chosen as $\mathbf{F}_t$
- ③ Relies on mean-variance theory in terms of tangency portfolio M .
 - The derivation of APT does not involve the tangency portfolio and mean-variance arguments
- ④ Considers only the equilibrium condition of the market
 - The multifactor model describes R_{it} instead of μ_i
- ⑤ Identification of the market portfolio is difficult
 - No tangency/market portfolio is needed in APT

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Fama-BacBeth Approach for Parameter Estimations

Multifactor model for asset $i \in \{1, \dots, n\}$: $R_{it} = a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it}$

APT: $\mu_i = \lambda_0 + \mathbf{b}_i^T \boldsymbol{\lambda}$

Unknown parameters: factor loadings $a_i, \mathbf{b}_i$ for $i \in \{1, \dots, n\}$; factor risk premium $\lambda_0, \boldsymbol{\lambda}$

Fama-MacBeth Approach is a two-step method for estimating $a_i, \mathbf{b}_i, \lambda_0, \boldsymbol{\lambda}$

1 Step 1, Time Series Regression (TSR):

- For each $i = 1, \dots, n$, regress $\{R_{it}\}_{t=1}^{\tau}$ against $\{\mathbf{F}_t\}_{t=1}^{\tau}$,

$$R_{it} = a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it}; \quad t = 1, \dots, \tau.$$

- Obtain estimated factor loadings $\hat{\mathbf{b}}_i, i = 1, \dots, n$

2 Step 2, Cross-Sectional Regression (CSR):

- For each $t = 1, \dots, \tau$, regress $\{R_{it}\}_{i=1}^n$ against $\{\hat{\mathbf{b}}_i\}_{i=1}^n$,

$$R_{it} = \lambda_{0t} + \hat{\mathbf{b}}_i^T \boldsymbol{\lambda}_t + \epsilon_{it}; \quad i = 1, \dots, n.$$

- Obtain estimated factor risk premiums $\hat{\lambda}_{0t}, \hat{\boldsymbol{\lambda}}_t, t = 1, \dots, \tau$
- Final Estimate:

$$\hat{\lambda}_0 = \frac{1}{\tau} \sum_{t=1}^{\tau} \hat{\lambda}_{0t}, \quad \text{and} \quad \hat{\boldsymbol{\lambda}} = \frac{1}{\tau} \sum_{t=1}^{\tau} \hat{\boldsymbol{\lambda}}_t.$$

Parameter Estimation: Time Series Regression

Multifactor model for asset $i \in \{1, \dots, n\}$: $R_{it} = a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it}$

• Unknown parameters: a_i and $\mathbf{b}_i$, $i \in \{1, \dots, n\}$

• Time-Series Regression (TSR) Approach for estimating $\{a_i\}$ and $\{\mathbf{b}_i\}$

• Time Series Data ($t = 1, \dots, \tau$) for a fixed asset i :

$$\bullet \mathbf{Y} = \begin{pmatrix} R_{i1} \\ \vdots \\ R_{i\tau} \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 & \mathbf{F}_1^T \\ \vdots & \vdots \\ 1 & \mathbf{F}_\tau^T \end{pmatrix} = \begin{pmatrix} 1 & F_{11} & F_{12} & \cdots & F_{1K} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & F_{\tau 1} & F_{\tau 2} & \cdots & F_{\tau K} \end{pmatrix}$$

• Regression Model:

$$\bullet \mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e} \Leftrightarrow \begin{pmatrix} R_{i1} \\ \vdots \\ R_{i\tau} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{F}_1^T \\ \vdots & \vdots \\ 1 & \mathbf{F}_\tau^T \end{pmatrix} \begin{pmatrix} a_i \\ \mathbf{b}_i \end{pmatrix} + \begin{pmatrix} e_{i1} \\ \vdots \\ e_{i\tau} \end{pmatrix}$$

• Estimation: $\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{a}_i \\ \hat{\mathbf{b}}_i \end{pmatrix} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}$

• Repeat for $i = 1, \dots, n \Rightarrow \{\hat{a}_1, \dots, \hat{a}_n\}, \{\hat{\mathbf{b}}_1, \dots, \hat{\mathbf{b}}_n\}$

Factor Risk Premium Estimation: Cross-Sectional Regression

APT: $\mu_i = \lambda_0 + \mathbf{b}_i^T \boldsymbol{\lambda}$

● Unknown parameters: factor risk premium $\lambda_0, \boldsymbol{\lambda}$

● Cross-Sectional Regression (CSR) Approach for estimating λ_0 and $\boldsymbol{\lambda}$

- Cross-Sectional Data ($i = 1, \dots, n$) for a fixed time t :

- $\mathbf{Y} = \begin{pmatrix} R_{1t} \\ \vdots \\ R_{nt} \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 & \hat{\mathbf{b}}_1^T \\ \vdots & \vdots \\ 1 & \hat{\mathbf{b}}_n^T \end{pmatrix} = \begin{pmatrix} 1 & \hat{b}_{11} & \hat{b}_{12} & \cdots & \hat{b}_{1K} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \hat{b}_{n1} & \hat{b}_{n2} & \cdots & \hat{b}_{nK} \end{pmatrix}$

- Regression Model:

- $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e} \Leftrightarrow \begin{pmatrix} R_{1t} \\ \vdots \\ R_{nt} \end{pmatrix} = \begin{pmatrix} 1 & \hat{\mathbf{b}}_1^T \\ \vdots & \vdots \\ 1 & \hat{\mathbf{b}}_n^T \end{pmatrix} \begin{pmatrix} \lambda_{0t} \\ \boldsymbol{\lambda}_t \end{pmatrix} + \begin{pmatrix} e_{1t} \\ \vdots \\ e_{nt} \end{pmatrix}$

- Estimation: $\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\lambda}_{0t} \\ \hat{\boldsymbol{\lambda}}_t \end{pmatrix} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}$

- Repeat for $i = 1, \dots, \tau \Rightarrow \{\hat{\lambda}_{01}, \dots, \hat{\lambda}_{0\tau}\}, \{\hat{\boldsymbol{\lambda}}_1, \dots, \hat{\boldsymbol{\lambda}}_\tau\}$

R Example: Fama-MacBeth for Fama-French 5 Factors

Fama-French 5: $R_{it} - r_t = a_i + b_i(R_{Mt} - r_t) + s_i SMB_t + h_i HML_t + r_i RMW_t + c_i CMA_t + e_{it}$

① **TSR:** $R_{it} = a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it} \quad (t = 1, \dots, \tau) \implies \{\hat{\mathbf{b}}_i\}_{i=1}^n$

② **CSR:** $R_{it} = \lambda_{0t} + \hat{\mathbf{b}}_i^T \boldsymbol{\lambda}_t + \epsilon_{it} \quad (i = 1, \dots, n) \implies \{\hat{\lambda}_{0t}, \hat{\boldsymbol{\lambda}}_t\}_{t=1}^\tau$

③ **Final Estimate:** $\hat{\lambda}_0 = \frac{1}{\tau} \sum_{t=1}^\tau \hat{\lambda}_{0t}$, and $\hat{\boldsymbol{\lambda}} = \frac{1}{\tau} \sum_{t=1}^\tau \hat{\boldsymbol{\lambda}}_t$

Example 2 (Fama-MacBeth Approach)

● Ch3FF5F.csv lists the stock returns (ReS) of six stocks and five factors

● Number of periods $\tau = 36$; Number of factors $K = 5$; Number of assets $n = 6$

Step 1: TSR for the 1st stock

```
dat <- read.csv("D:/Ch3FF5F.csv") # Read file
summary(lm(ReS1~ReM+SMB+HML+RMW+CMA, data=dat)) # Regression
```

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	0.0009085	0.0108788	0.084	0.93400	
ReM	0.7322980	0.3315143	2.209	0.03496	*
SMB	-0.6738742	0.9927284	-0.679	0.50246	
HML	2.3118012	1.0274540	2.250	0.03194	*
RMW	3.6892253	1.5810800	2.333	0.02652	*
CMA	-4.8841095	1.7186760	-2.842	0.00799	**

Result:

$$R_{it} - r_t = 0.00091 + 0.73(R_{Mt} - r_t) - 0.67SMB_t + 2.31HML_t + 3.69RMW_t - 4.88CMA_t + e_{it}$$

R Example: Fama-MacBeth for Fama-French 5 Factors

Fama-French 5: $R_{it} - r_t = a_i + b_i(R_{Mt} - r_t) + s_i SMB_t + h_i HML_t + r_i RMW_t + c_i CMA_t + e_{it}$

① **TSR:** $R_{it} = a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it} \quad (t = 1, \dots, \tau) \implies \{\hat{\mathbf{b}}_i\}_{i=1}^n$

② **CSR:** $R_{it} = \lambda_{0t} + \hat{\mathbf{b}}_i^T \boldsymbol{\lambda}_t + \epsilon_{it} \quad (i = 1, \dots, n) \implies \{\hat{\lambda}_{0t}, \hat{\boldsymbol{\lambda}}_t\}_{t=1}^\tau$

● **Final Estimate:** $\hat{\lambda}_0 = \frac{1}{\tau} \sum_{t=1}^\tau \hat{\lambda}_{0t}$, and $\hat{\boldsymbol{\lambda}} = \frac{1}{\tau} \sum_{t=1}^\tau \hat{\boldsymbol{\lambda}}_t$

Example 3 (Fama-MacBeth Approach)

● Number of periods $\tau = 36$; Number of factors $K = 5$; Number of assets $n = 6$

Step ①: TSR

```
b=matrix(0,6,5) # n x K = 6x5 empty matrix to store  $\hat{\mathbf{b}}_1, \dots, \hat{\mathbf{b}}_6$ 
for (i in 1:6){ # Stock returns are from columns 7-12
  b[i,]=lm(dat[,6+i]~ReM+SMB+HML+RMW+CMA, data=dat)$coef[-1] #store  $\hat{\mathbf{b}}_i$ 
}
```

Step ②: CSR

```
res=matrix(0,36,6) #  $\tau \times (K+1) = 36 \times 6$  empty matrix to store lambdas
for (t in 1:36){
  temp=as.numeric(dat[t,7:12]) # get all  $n$  returns on day  $t$ 
  res[t,]=lm(temp~b)$coef #store estimated lambdas for day  $t$ 
}
```

● **Final Estimates:**

```
lambda0=mean(res[,1]) # obtain  $\lambda_0$ 
lambda=apply(res[,2:6],2,mean) # obtain  $\boldsymbol{\lambda}$  by taking column mean
```

Agenda

- 1 Factor Models
 - Multifactor Model
 - Choice of Factors
 - Fama-French Factor Models
- 2 Arbitrage Pricing Theory (APT)
- 3 Estimation
 - Time Series Regression
 - Cross-Sectional Regression
 - R Application
- 4 Summary
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Summary

- **Multifactor model:** $R_t = \mathbf{a} + \mathbf{B}\mathbf{F}_t + \mathbf{e}_t$, $R_t, \mathbf{a}_t, \mathbf{e}_t \in \mathbb{R}^n$, $\mathbf{B} \in \mathbb{R}^{n \times K}$, $\mathbf{F}_t \in \mathbb{R}^K$
 - $\mathbb{E}(e_{it}) = 0$, $\text{Var}(e_{it}) = \sigma_{ei}^2$, $\mathbb{E}(e_{it}e_{jt}) = 0$, $\text{Cov}(e_{it}, \mathbf{F}_t) = \mathbf{0}$
 - For a **portfolio** $\mathbf{w} \in \mathbb{R}^n$ with return $R_{pt} = \mathbf{w}^T \mathbf{a} + \mathbf{w}^T \mathbf{B}\mathbf{F}_t + \mathbf{w}^T \mathbf{e}_t$:
 - **Expected Return:** $\mu_p = \mathbf{w}^T \mathbf{a} + \mathbf{w}^T \mathbf{B}\boldsymbol{\mu}_F$, $\boldsymbol{\mu}_F = \mathbb{E}(\mathbf{F}_t)$
 - **Variance:** $\sigma_p^2 = \mathbf{w}^T \mathbf{B}\boldsymbol{\Omega}_F \mathbf{B}^T \mathbf{w} + \mathbf{w}^T \boldsymbol{\Psi}_e \mathbf{w}$, $\boldsymbol{\Omega}_F = \text{Cov}(\mathbf{F}_t)$, $\boldsymbol{\Psi}_e = \text{Diag}(\sigma_{ei}^2)$
- **Fama-French 3-factor model:**
$$R_{it} - r_t = a_i + b_i(R_{Mt} - r_t) + s_i \text{SMB}_t + h_i \text{HML}_t + e_{it}$$
 - **SMB:** Small-Big (market cap), **HML:** High-Low (Book to market equity)
- **Fama-French 5-factor model:**
$$R_{it} - r_t = a_i + b_i(R_{Mt} - r_t) + s_i \text{SMB}_t + h_i \text{HML}_t + r_i \text{RMW}_t + c_i \text{CMA}_t + e_{it}$$
 - **RMW:** Robust-Weak (profitability), **CMA:** Conservative-Aggressive (investment)
- **APT:** $\mu_i = \lambda_0 + \mathbf{b}_i^T \boldsymbol{\lambda}$ under the K -factor model $R_i = a_i + \mathbf{b}_i^T \mathbf{F} + e_i$; $i = 1, \dots, n$
 - $\lambda_0 = r$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_K)$ are **factor risk premiums**
 - $\lambda_i = \mathbf{w}_i^T (\boldsymbol{\mu} - r\mathbf{1})$ where $\mathbf{w}_i^T \mathbf{R} = \mathbf{w}_i^T \mathbf{a} + F_i + \mathbf{w}_i^T \mathbf{e}$ (**Arrow-Debreu security**)
- **Fama-MacBeth Approach** for estimating factor loadings and factor risk premiums
 - 1 **TSR:** $R_{it} = a_i + \mathbf{b}_i^T \mathbf{F}_t + e_{it}$ ($t = 1, \dots, \tau$) $\implies \{\hat{\mathbf{b}}_i\}_{i=1}^n$
 - 2 **CSR:** $R_{it} = \lambda_{0t} + \hat{\mathbf{b}}_i^T \boldsymbol{\lambda}_t + \epsilon_{it}$ ($i = 1, \dots, n$) $\implies \{\hat{\lambda}_{0t}, \hat{\boldsymbol{\lambda}}_t\}_{t=1}^\tau$
 - **Final Estimate:** $\hat{\lambda}_0 = \frac{1}{\tau} \sum_{t=1}^\tau \hat{\lambda}_{0t}$, and $\hat{\boldsymbol{\lambda}} = \frac{1}{\tau} \sum_{t=1}^\tau \hat{\boldsymbol{\lambda}}_t$

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