

Continuous Time Financial Model

Chapter 5: Continuous Time Financial Model

Review: n -period Binomial Model

- Time 0: $(B_0, S_0) = (1, s)$

- Time $t_k = k\delta$: $B_k = e^{rt_k}$, $S_k = \begin{cases} S_{k-1}u & \text{if market goes up} \\ S_{k-1}d & \text{if market goes down} \end{cases}$

- State price vector at each branch $\left[\text{Using } \mathbf{S}^0 = \begin{pmatrix} e^{rt_k} \\ S_k \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} e^{rt_{k+1}} & S_k u \\ e^{rt_{k+1}} & S_k d \end{pmatrix} \right]$:

$$\psi' = (\mathbf{S}^0)' \mathbf{S}^{-1} = \frac{1}{e^{r\delta}(u-d)} \begin{pmatrix} e^{r\delta} - d & u - e^{r\delta} \end{pmatrix} = (\psi_u, \psi_d)$$

- Path $\omega_t = (Z_1, \dots, Z_t)'$ with $Z_k = \begin{cases} 1 & \text{if market goes up,} \\ -1 & \text{if market goes down.} \end{cases}$

- State Price Vector associated with the path $\omega_n^j = (Z_1^j, \dots, Z_n^j)'$; ($j = 1, \dots, 2^n$):

$$\Psi^j = \prod_{t=1}^n \left(\psi_u \mathbb{I}_{\{Z_t^j=1\}} + \psi_d \mathbb{I}_{\{Z_t^j=-1\}} \right)$$

- Price of $F = (F^1, \dots, F^{2^n})'$ at $t = 0$: $F^0 = \sum_{j=1}^{2^n} \Psi^j F^j$

- Idea: Define a continuous model by taking a limit of multi-period binomial model with many ($n \rightarrow \infty$) small ($\delta \rightarrow 0$) time periods, with $n\delta = T$.

Agenda

1 Extensions to Continuous Time Financial Model

- Distribution of S_T under risk neutral measure

2 Pricing under Continuous Time Model

3 Brownian Motions and Stochastic Calculus

From Multi-period to Continuous Time Model

Goal: Multi-period binomial model with many ($n \rightarrow \infty$) small ($\delta \rightarrow 0$) time periods, $n\delta = T$.

- 1 Partition $[0, T]$ into n intervals of length δ .
- 2 Bond price at time $t = k\delta$: $B_{k\delta} = e^{rk\delta}$.
- 3 Stock price at time $t = k\delta$ is modeled by a binomial tree: (S_0 fixed)

$$S_{k\delta} = S_{(k-1)\delta} \exp \left\{ \mu\delta + \sigma\sqrt{\delta}Z_k \right\}, \quad \text{where } Z_k = \begin{cases} 1 & \text{if up,} \\ -1 & \text{if down.} \end{cases}$$

- $u = \exp(\mu\delta + \sigma\sqrt{\delta})$ and $d = \exp(\mu\delta - \sigma\sqrt{\delta})$
- stock grows with rate $\exp(\mu\delta)$ with volatility $\exp(\pm\sigma\sqrt{\delta})$
- **Why $\sqrt{\delta}$? If $E(Z_k) = 0$ and $\text{Var}(Z_k) = 1$, then by CLT:**

$$S_T = S_{n\delta} = S_0 e^{\mu n\delta + \sigma\sqrt{n\delta} \frac{1}{\sqrt{n}} \sum_{k=1}^n Z_k} \xrightarrow{CLT} S_0 e^{\mu T + \sigma\sqrt{T} N(0,1)} \text{ limit exists!}$$

- 4 Pricing $F^0 = \sum_{j=1}^{2^n} \Psi^j F^j$

- Discrete Version:

- The state prices $\{\Psi^j\}$ correspond to **point mass** on 2^n events of S_T

- Continuous version:

- The state prices Ψ correspond to a **distribution of continuous r.v.** S_T

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Continuous Time Model: Distribution of S_T

Model: $S_{k\delta} = S_{(k-1)\delta} \exp \{ \mu\delta + \sigma\sqrt{\delta}Z_k \}$, where $Z_k = \begin{cases} 1 & \text{if up,} \\ -1 & \text{if down.} \end{cases}$

State price vector at each branch: $(\psi_u, \psi_d) = \frac{1}{e^{r\delta}(u-d)} \begin{pmatrix} e^{r\delta} - d, & u - e^{r\delta} \end{pmatrix}$

Distribution of S_T :

- At time $T = n\delta$ (the end of the n -th period),

$$S_T = S_0 \exp \left\{ \mu n\delta + \sigma\sqrt{\delta} \sum_{k=1}^n Z_k \right\}.$$

- Randomness of S_T is solely from $\sum_{k=1}^n Z_k$
- Physical probability for the stock to go up or down is not important.
- Risk neutral prob. (before discounting) of going up and down are

$$\text{up : } \psi_u e^{r\delta} \triangleq P(Z_k = 1) = \frac{e^{r\delta} - d}{u - d} = \frac{e^{r\delta} - e^{\mu\delta - \sigma\sqrt{\delta}}}{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{\mu\delta - \sigma\sqrt{\delta}}} \triangleq p,$$

$$\text{down : } \psi_d e^{r\delta} \triangleq P(Z_k = -1) = 1 - p.$$

Continuous Time Model: Distribution of S_T

Model: $S_T = S_0 \exp \left\{ \mu n \delta + \sigma \sqrt{\delta} \sum_{k=1}^n Z_k \right\}, \quad Z_k = \begin{cases} 1 & \text{if up}(p), \\ -1 & \text{if down}(1-p). \end{cases}$

Basic properties of Bernoulli random variable Z_t :

$$\begin{aligned} \mathbb{E}(Z_k) &= p - (1-p) = 2p - 1, \\ \text{Var}(Z_k) &= p + (1-p) - (2p-1)^2 = 4p(1-p). \end{aligned}$$

Normalizing $\sum_{k=1}^n Z_k$ by its mean and standard deviation:

$$\begin{aligned} S_T &= S_0 \exp \left\{ \mu n \delta + \sigma \sqrt{\delta} \left(\sum_{k=1}^n \frac{Z_k - (2p-1)}{\sqrt{4p(1-p)}} \sqrt{4p(1-p)} + n(2p-1) \right) \right\} \\ &= S_0 \exp \left\{ \mu n \delta + \sigma n \delta \frac{2p-1}{\sqrt{\delta}} + \sigma \sqrt{4\delta p(1-p)} \sum_{k=1}^n Y_k \right\}, \end{aligned}$$

where $Y_k \triangleq \frac{Z_k - (2p-1)}{\sqrt{4p(1-p)}}$ is a random variable with mean 0 & variance 1.

Continuous Time Model: Distribution of S_T

Model: $S_T = S_0 \exp \left\{ \mu n \delta + \sigma n \delta \frac{2p-1}{\sqrt{\delta}} + \sigma \sqrt{4\delta p(1-p)} \sum_{k=1}^n Y_k \right\}$

- By construction, $n\delta = T \Rightarrow \delta = \frac{T}{n}$, so

$$S_T = S_0 \exp \left\{ \mu T + \sigma T \frac{2p-1}{\sqrt{\delta}} + \sigma \sqrt{4Tp(1-p)} \frac{\sum_{k=1}^n Y_k}{\sqrt{n}} \right\},$$

- Using a **second order** Taylor's expansion $e^x = 1 + x + \frac{x^2}{2} + \dots$,

$$\begin{aligned} p &= \frac{e^{r\delta} - e^{\mu\delta - \sigma\sqrt{\delta}}}{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{\mu\delta - \sigma\sqrt{\delta}}} \approx \frac{(1 + r\delta) - (1 + \mu\delta - \sigma\sqrt{\delta} + \frac{1}{2}\sigma^2\delta)}{(1 + \mu\delta + \sigma\sqrt{\delta} + \frac{1}{2}\sigma^2\delta) - (1 + \mu\delta - \sigma\sqrt{\delta} + \frac{1}{2}\sigma^2\delta)} \\ &= \frac{\sqrt{\delta}(r - \mu - \frac{1}{2}\sigma^2)}{2\sigma} + \frac{1}{2} \end{aligned}$$

- As $\delta \rightarrow 0$,

① $p \rightarrow \frac{1}{2}$ ② $\frac{2p-1}{\sqrt{\delta}} \rightarrow -\frac{1}{\sigma} \left(\mu + \frac{1}{2}\sigma^2 - r \right)$

③ **Central Limit Theorem:** $\frac{\sum_{k=1}^n Y_k}{\sqrt{n}} \xrightarrow{d} N(0, 1)$

Black-Scholes Model

Multi-period Binomial Model: $S_{k\delta} = S_{(k-1)\delta} \exp \left\{ \mu\delta + \sigma\sqrt{\delta}Z_k \right\}$; $Z_k = \begin{cases} 1 & \text{if up,} \\ -1 & \text{if down.} \end{cases}$

- **Model:** $S_T = S_0 \exp \left\{ \mu T + \sigma T \frac{2p-1}{\sqrt{\delta}} + \sigma \sqrt{4Tp(1-p)} \frac{\sum_{k=1}^n Y_k}{\sqrt{n}} \right\}$.
- **Limits:** $p \rightarrow \frac{1}{2}$, $\frac{2p-1}{\sqrt{\delta}} \rightarrow -\frac{1}{\sigma} \left(\mu + \frac{1}{2}\sigma^2 - r \right)$, $\frac{\sum_{k=1}^n Y_k}{\sqrt{n}} \xrightarrow{d} N(0, 1)$
- We have proved the following theorem:

Theorem 1

For the multi-period Binomial model with $n \rightarrow \infty$, $\delta \rightarrow 0$ and $n\delta = T$, the distribution of S_T under the risk neutral measure is

$$S_T \xrightarrow{d} S_0 \exp \left\{ \left(r - \frac{1}{2}\sigma^2 \right) T + \sigma N(0, T) \right\}.$$

We call it the **Continuous Time Pricing Model** or **Black-Scholes Model**.

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Pricing under Continuous Time Model

$$\text{Black-Scholes Model: } S_T = S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma N(0, T) \right\}$$

Theorem 2 (Pricing under the Black-Schole Model)

If a financial derivative F has a payoff $F_T = f(S_T)$, then the price F^0 is

$$F^0 = e^{-rT} \int f \left(S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma x \right\} \right) \phi_T(x) dx,$$

where $\phi_T(x)$ is the density of the Normal distribution $N(0, T)$.

Proof.

- The price (before discounting) of F with payoff $F_T = f(S_T)$ is

$$\mathbb{E}_{\tilde{\Psi}}(f(S_T)) := \int f \left(S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma x \right\} \right) \phi_T(x) dx,$$

- Price at time 0 is $F^0 = e^{-rT} \mathbb{E}_{\tilde{\Psi}}(f(S_T))$.

Black-Scholes' Call Option Price

Example 3 (Black and Scholes Call Option Formula (1973))

- European Call option with strike price K has payoff $f(S_T) = (S_T - K)^+$
- Price of the Call option: $\left[\text{using } F^0 = e^{-rT} \mathbb{E}_{\tilde{\Psi}}(f(S_T)) \text{ \& } S_T = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma N(0,T)} \right]$

$$\begin{aligned} F^0 &= e^{-rT} \int \left(S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma x \right\} - K \right)^+ \phi_T(x) dx \\ &= e^{-rT} \int \left(S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} z \right\} - K \right)^+ \phi(z) dz \quad \left(z = \frac{x}{\sqrt{T}} \right) \\ &= \int_{-d_2}^{\infty} \left(S_0 \exp \left\{ -\frac{1}{2} \sigma^2 T + \sigma \sqrt{T} z \right\} - K e^{-rT} \right) \phi(z) dz \end{aligned}$$

$$\left(S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} z \right\} - K \geq 0 \Rightarrow z \geq - \left[\ln(S_0/K) + \left(r - \frac{1}{2} \sigma^2 \right) T \right] / \sigma \sqrt{T} =: -d_2 \right)$$

$$\begin{aligned} F^0 &= S_0 \int_{-d_2 - \sigma \sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - \sigma \sqrt{T})^2} d(z - \sigma \sqrt{T}) - K e^{-rT} \int_{-d_2}^{\infty} \phi(z) dz \\ &= S_0 N(d_1) - K e^{-rT} N(d_2) \end{aligned}$$

- $N(\cdot)$ is the cumulative distribution function of $N(0, 1)$.
- $d_2 = \left[\ln \left(\frac{S_0}{K} \right) + \left(r - \sigma^2/2 \right) T \right] / \sigma \sqrt{T}$; $d_1 = d_2 + \sigma \sqrt{T}$

Pricing: Digital Call Option, or Binary Call Option

Black-Scholes Model: $S_T = S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma N(0, T) \right\}$

Price of derivative: $F^0 = e^{-rT} \mathbb{E}_{\tilde{\Psi}}(f(S_T))$

Example 4 (Digital/Binary Call Option)

- Digital Call option with strike price K has payoff $f(S_T) = 1_{\{S_T \geq K\}}$
- Price of the Call option:

$$\begin{aligned} F^0 &= e^{-rT} \mathbb{E}_{\tilde{\Psi}}(1_{\{S_T \geq K\}}) \\ &= e^{-rT} \mathbb{P}_{\tilde{\Psi}}(\{S_T \geq K\}) \\ &= e^{-rT} \mathbb{P} \left\{ S_0 \exp \left[\left(r - \frac{1}{2} \sigma^2 \right) T + \sigma N(0, T) \right] \geq K \right\} \\ &= e^{-rT} \mathbb{P} \left\{ \frac{N(0, T)}{\sqrt{T}} \geq -\frac{\ln(S_0/K) + (r - \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \right\} \\ &= e^{-rT} N(d_2) \end{aligned}$$

- $N(\cdot)$ is the cumulative distribution function of $N(0, 1)$.
- $d_2 = \left[\ln \left(\frac{S_0}{K} \right) + (r - \sigma^2/2)T \right] / \sigma \sqrt{T}$;

Valuation of Derivatives

A new method to determine the value of derivatives



My principal contribution to the Black-Scholes option-pricing theory was to show that the dynamic trading strategy prescribed by Black and Scholes to offset the risk exposure of an option would provide a perfect hedge in the limit of continuous trading.

— *Robert Merton* —

Robert Merton (1944~): Nobel Economic Sciences 1997

Valuation of Derivatives

A new method to determine the value of derivatives



All models have faults - that doesn't mean you can't use them as tools for making decisions.

— Myron Scholes —

AZ QUOTES

Myron Scholes (1941~): Nobel Economic Sciences 1997

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Brownian Motions and Stochastic Calculus

- For European call, payoff $(S_T - K)^+$ depends only on S_T

⇒ Price = $\bar{\Psi} E_{\tilde{\Psi}}(f(S_T))$ involves a single integral

- For general options such as

- Up-and-in barrier call option: payoff = $(S_T - K)^+ 1_{\{\max_{t \in [0, T]} S_t \geq B\}}$,
- Asian call option: payoff = $\left(\int_0^T S_t dt - K\right)^+$,

the **payoff depends on the whole path** $\{S_t\}_{t \in [0, T]}$

⇒ Price = $\bar{\Psi} E_{\tilde{\Psi}}\left(f\left(\{S_t\}_{t \in [0, T]}\right)\right)$ is difficult to compute analytically

- **Solution**

- Describe the dynamics of $\{S_t\}_{t \in [0, T]}$ as a stochastic process
([Stochastic Calculus here](#))
- Based on the dynamics of $\{S_t\}_{t \in [0, T]}$, simulate sample paths to approximate $\bar{\Psi} E_{\tilde{\Psi}}\left(f\left(\{S_t\}_{t \in [0, T]}\right)\right)$ ([see RMSC 4001](#))

Modeling the stochastic process $\{S_t\}_{t \in [0, T]}$

n -period Model: $S_T = S_0 \exp \left\{ \mu n \delta + \sigma \sqrt{\delta} \sum_{k=1}^n Z_k \right\}$, $Z_k = \begin{cases} 1 & \text{if up}(p), \\ -1 & \text{if down}(1-p). \end{cases}$

Black-Scholes Model: $S_T = S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma N(0, T) \right\}$ ①

- The proof of ① readily extends to a general t instead of the maturity T :

$$S_t = S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}$$

- **Brownian Motion $\{W_t\}$:**
 - $W_t \sim N(0, t)$ for each $t \in [0, \infty)$
 - is the limit (after rescaling) of partial sum $\sum_{k=1}^{[nt]} Z_k$ of i.i.d. r.v.s
 - is continuous but not differentiable!
- Black-Scholes model for $\{S_t\}$ is simple and has closed-form expression
 - Advanced models without closed-forms exist as **SDEs**
 - **Ornstein-Uhlenbeck Model:** $dS_t = \beta S_t dt + \sigma dW_t$
 - **Cox-Ingersoll-Ross (CIR) Model:** $dS_t = (\alpha - \beta S_t) dt + \sigma \sqrt{S_t} dW_t$
 - **Vasicek Model:** $dS_t = (\alpha + \beta S_t) dt + \sigma dW_t$
 - **Ho-Lee Model:** $dS_t = a(t) dt + \sigma dW_t \dots \dots$ (more in RMSC 4007)

⇒ Need **Stochastic Calculus** to manipulate these SDEs!

Summary

- The ***n*-period binomial model**: $n\delta = T$
 - $S_T = S_0 \exp \left\{ \mu n\delta + \sigma \sqrt{\delta} \sum_{k=1}^n Z_k \right\}$, $Z_k \begin{cases} 1 & \text{if up,} \\ -1 & \text{if down.} \end{cases}$
 - Risk Neutral Probability at each branch (before discounting):
up: $\psi_u e^{r\delta} \triangleq P(Z_k = 1) = \frac{e^{r\delta} - d}{u - d}$; **down**: $\psi_d e^{r\delta} \triangleq P(Z_k = -1) = \frac{u - e^{r\delta}}{u - d}$
 - $u = \exp(\mu\delta + \sigma\sqrt{\delta})$ & $d = \exp(\mu\delta - \sigma\sqrt{\delta})$
- The **Black Scholes model**: Taking limit $n \rightarrow \infty$, $\delta \rightarrow 0$, $T = n\delta$
 - $S_T \xrightarrow{d} S_0 \exp \left\{ \left(r - \frac{1}{2}\sigma^2 \right) T + \sigma N(0, T) \right\}$, under risk neutral measure
- **Pricing formula of BS model**: Price of derivative payoff $F_T = f(S_T)$ is

$$F^0 = \bar{\Psi} E_{\tilde{\Psi}}(f(S_T)) = e^{-rT} \int f \left(S_0 \exp \left\{ \left(r - \frac{1}{2}\sigma^2 \right) T + \sigma x \right\} \right) \phi_T(x) dx$$

- **Black Scholes pricing formula for European call**:

$$F^0 = S_0 N(d_1) - K e^{-rT} N(d_2)$$

- $N(\cdot)$ is CDF of $N(0, 1)$
- $d_2 = \left[\ln \left(\frac{S_0}{K} \right) + (r - \sigma^2/2)T \right] / \sigma\sqrt{T}$
- $d_1 = d_2 + \sigma\sqrt{T}$