

Stochastic Calculus

Chapter 6: Stochastic Calculus

Agenda

- 1 Functional Central Limit Theorem
- 2 Brownian Motion and Wiener Process
- 3 Stochastic Integral
 - Properties of Stochastic Integral
- 4 Itô's Lemma
- 5 Stochastic Product Rule and Integration by Parts
- 6 Black-Scholes PDE

Two Fundamental Theorems in Probability Theory

Theorem 1 (Central Limit Theorem)

If $\{Y_j\}$ $\overset{i.i.d.}{\sim}$ mean 0 & variance 1, then as $n \rightarrow \infty$,

$$\frac{\sum_{j=1}^n Y_j}{\sqrt{n}} \xrightarrow{d} N(0, 1).$$

- $\xrightarrow{d}$ denotes *convergence in distribution*. (See STAT 4003 or RMSC 4005)

Theorem 2 (Functional Central Limit Theorem)

- Suppose that $\{Y_j\}$ $\overset{i.i.d.}{\sim}$ mean 0 & variance 1.
- Define the *partial sum process* $S_Y(t) = \sum_{j=1}^{[nt]} Y_j$.
 - a *random* function mapping from $t \in (0, \infty)$ to $\mathbb{R}$
- As $n \rightarrow \infty$,

$$\frac{S_Y(t)}{\sqrt{n}} = \frac{\sum_{j=1}^{[nt]} Y_j}{\sqrt{n}} \Rightarrow W_t,$$

- W_t is Brownian Motion (or Wiener process) satisfying $W_t \sim N(0, t)$
- $\Rightarrow$ denotes *weak convergence of stochastic processes*. (See STAT 5005)

Brownian Motion $\{W_t\}$ in asset modeling

n -period Binomial Model: $S_{k\delta} = S_{(k-1)\delta} \exp \left\{ \mu\delta + \sigma\sqrt{\delta}Z_k \right\}$; $Z_k = \begin{cases} 1 & \text{up } (p) \\ -1 & \text{down } (1-p) \end{cases}$

• $T = n\delta$, $p \rightarrow \frac{1}{2}$, $\frac{2p-1}{\sqrt{\delta}} \rightarrow -\frac{1}{\sigma} \left(\mu + \frac{1}{2}\sigma^2 - r \right)$, $Y_k \triangleq \frac{Z_k - (2p-1)}{\sqrt{4p(1-p)}}$ mean 0, variance 1

Black Scholes Model: $S_T = S_0 \exp \left\{ \left(r - \frac{1}{2}\sigma^2 \right) T + \sigma N(0, T) \right\}$

Functional CLT: If $Y_j \stackrel{d}{\sim}$ (mean 0, variance 1), then $\sum_{k=1}^{[nt]} \frac{Y_k}{\sqrt{n}} \Rightarrow W_t$, where $W_t \sim N(0, t)$

Extension to model S_t at a general time point t instead of maturity T

• For $t \in [0, T]$, let $k = \left[n \frac{t}{T} \right]$ (the biggest integer less than $n \frac{t}{T}$)

• $k \leq n \frac{t}{T} < k+1 \Rightarrow k\delta \leq t < (k+1)\delta$

• As $n \rightarrow \infty$, $\delta \rightarrow 0$ and $n\delta = T$, we have

$$\begin{aligned} S_t &= S_0 e^{\mu k\delta + \sigma\sqrt{\delta} \sum_{j=1}^k Z_j} = S_0 e^{\mu k\delta + \sigma\sqrt{\delta} \left(\sum_{j=1}^k \frac{Z_j - (2p-1)}{\sqrt{4p(1-p)}} \sqrt{4p(1-p)} + k(2p-1) \right)} \\ &= S_0 e^{\mu k\delta + \sigma k\delta \frac{2p-1}{\sqrt{\delta}} + \sigma\sqrt{4Tp(1-p)} \frac{1}{\sqrt{n}} \sum_{j=1}^{\left[\frac{nt}{T} \right]} Y_j} \\ &\Rightarrow S_0 e^{\left(r - \frac{1}{2}\sigma^2 \right)t + \sqrt{T}\sigma W_{\frac{t}{T}}} \quad (FCLT) \\ &= S_0 e^{\left(r - \frac{1}{2}\sigma^2 \right)t + \sigma W_t} \quad \left(\sqrt{T}W_{\frac{t}{T}} \sim N \left(0, T \frac{t}{T} \right) = N(0, t) \sim W_t \right) \end{aligned}$$

• The BS model $S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2 \right)t + \sigma W_t}$ is also called **Geometric Brownian Motion**.

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Brownian Motion and Wiener Process $\{W_t\}$

Functional CLT: If $Y_j \stackrel{d}{\sim}$ (mean 0, variance 1), then $\frac{\sum_{k=1}^{\lfloor nt \rfloor} Y_k}{\sqrt{n}} \Rightarrow W_t$, where $W_t \sim N(0, t)$

Definition 1 (Brownian Motion or Wiener Process)

A Brownian Motion (or Wiener process) is a stochastic process satisfying:

- **Stationary increment:** For $s < t$, $W_t - W_s \sim N(0, t - s)$
- **Independent increment:** For $0 \leq t_1 < t_2 \leq t_3 < t_4$,

$W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent random variables .

- **Starts at 0:** $\mathbb{P}(\{W_0 = 0\}) = 1$.

History:

- Botanist Robert Brown (1773-1858) first described the phenomenon in 1827 while looking through a microscope at pollen of plant in water
- Mathematician Norbert Wiener (1894-1964) developed the mathematical theory of Brownian motion

Properties of Brownian Motions

Brownian Motion:

- ① $W_t - W_s \sim N(0, t-s)$ ② $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ independent ③ $\mathbb{P}(\{W_0 = 0\}) = 1$

Properties (of Brownian Motion)

① **Normally distributed:** $W_t \sim N(0, t), \quad \forall t$

② **Martingale:** for $s < t$, $\mathbb{E}(W_t | W_s) = W_s$

Proof: $\mathbb{E}(W_t | W_s) = \mathbb{E}(W_t - W_s + W_s | W_s) \stackrel{① ②}{=} E(N(0, t-s)) + E(W_s | W_s) = W_s$

③ **Continuity:** W_t is a continuous function of t

Intuition: $W_{t+\Delta} - W_t \stackrel{①}{\sim} N(0, \Delta) \rightarrow 0$ as $\Delta \rightarrow 0$

④ **Non-differentiability:** W_t is not differentiable $\forall t$

Intuition: $W_{t+\Delta} - W_t \stackrel{①}{\sim} N(0, \Delta), \forall t$ implies that the process wiggles

⑤ **Covariance:** $\text{Cov}(W_s, W_t) = \min(s, t)$

Proof: If $s < t$, $\text{Cov}(W_s, W_t) = \text{Cov}(W_s, W_s) + \text{Cov}(W_s, W_t - W_s) = \text{Var}(W_s) + 0 = s$

If $t \leq s$, $\text{Cov}(W_s, W_t) = \text{Cov}(W_t, W_t) + \text{Cov}(W_s - W_t, W_t) = 0 + \text{Var}(W_t) = t$

Brownian Motions: Examples

Brownian Motion:

- ① $W_t - W_s \sim N(0, t-s)$ ② $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ independent ③ $\mathbb{P}(\{W_0 = 0\}) = 1$

Example 1 (Verifying Brownian Motions)

① (Scaling): Show that $\tilde{W}_t = \frac{1}{\sqrt{c}} W_{ct}$ is Brownian motion, where $c \in \mathbb{R}$.

① : Obviously, $\tilde{W}_t - \tilde{W}_s$ is normal with mean 0.

$$\text{Var}(\tilde{W}_t - \tilde{W}_s) = \frac{1}{c} \text{Var}(W_{ct} - W_{cs}) = \frac{1}{c} c(t-s) = t-s$$

② : As $ct_1 < ct_2 \leq ct_3 < ct_4$, $W_{ct_4} - W_{ct_3}$ and $W_{ct_2} - W_{ct_1}$ are independent $\Rightarrow$ Done

② (Time inversion): Show that $\tilde{W}_t = tW_{1/t}$ is a Brownian Motion.

① : $\tilde{W}_t - \tilde{W}_s = t(W_{1/t} - W_{1/s}) + (t-s)W_{1/s}$ is normal with mean 0.

$$\text{Var}(\tilde{W}_t - \tilde{W}_s) = \text{Cov}(\tilde{W}_t - \tilde{W}_s, \tilde{W}_t - \tilde{W}_s) = t^2 t^{-1} - 2st \min(t^{-1}, s^{-1}) + s^2 s^{-1} = t-s$$

② : $\text{Cov}(\tilde{W}_{t_4} - \tilde{W}_{t_3}, \tilde{W}_{t_2} - \tilde{W}_{t_1}) = \text{Cov}(t_4 W_{1/t_4} - t_3 W_{1/t_3}, t_2 W_{1/t_2} - t_1 W_{1/t_1})$
 $= t_2 t_4 / t_4 - t_2 t_3 / t_3 - t_1 t_4 / t_4 + t_1 t_3 / t_3 = 0$, where $t_1 < t_2 \leq t_3 < t_4$

③ (Time reversal): Show that $\tilde{W}_t = W_1 - W_{1-t}, \forall 0 \leq t \leq 1$ is Brownian Mo.

① : $\tilde{W}_t - \tilde{W}_s = W_{1-s} - W_{1-t}$ is normal with mean 0.

$$\text{Var}(\tilde{W}_t - \tilde{W}_s) = \text{Var}(W_{1-s} - W_{1-t}) = (1-s) - (1-t) = t-s$$

② : $\text{Cov}(\tilde{W}_{t_4} - \tilde{W}_{t_3}, \tilde{W}_{t_2} - \tilde{W}_{t_1}) = \text{Cov}(W_{1-t_3} - W_{1-t_4}, W_{1-t_1} - W_{1-t_2})$
 $= (1-t_3) - (1-t_4) - (1-t_3) + (1-t_4) = 0$

Brownian Motions: Examples

Brownian Motion:

- ① $W_t - W_s \sim N(0, t-s)$ ② $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ independent ③ $\mathbb{P}(\{W_0 = 0\}) = 1$

Example 2 (Probability related to Brownian Motions)

- ① Find $\mathbb{P}(W_t - W_0 > \sqrt{t})$

Ans: $\mathbb{P}(W_t - W_0 > \sqrt{t}) = \mathbb{P}(N(0, t) > \sqrt{t}) = \mathbb{P}(N(0, 1) > 1) = 1 - \Phi(1) \approx 0.16$

- ② Find $\mathbb{P}(W_t < a | W_s)$

Ans: $\mathbb{P}(W_t < a | W_s) = \mathbb{P}(W_t - W_s + W_s < a | W_s) \stackrel{②}{=} \mathbb{P}(N(0, t-s) < a - W_s | W_s)$
 $= \Phi\left(\frac{a - W_s}{\sqrt{t-s}}\right)$

Brownian Motions: Examples

Brownian Motion:

- ① $W_t - W_s \sim N(0, t-s)$ ② $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ independent ③ $\mathbb{P}(\{W_0 = 0\}) = 1$

Example 3 (Quadratic Variation (QV) of Brownian Motions)

- ① Let $Q_n = \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2$, where $t_i = \frac{i}{n}T$. Find $\lim_{n \rightarrow \infty} Q_n$.

Ans: From ① and ②, $W_{t_{i+1}} - W_{t_i} \sim N(0, T/n)$ are independent across i .

Let $Z_i \stackrel{iid}{\sim} N(0, 1)$, we can express Q_n as

$$\begin{aligned} Q_n &= \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 \stackrel{D}{=} \sum_{i=0}^{n-1} \left(\sqrt{\frac{T}{n}} Z_i \right)^2 = T \frac{\sum_{i=0}^{n-1} Z_i^2}{n} \\ &\xrightarrow{n \rightarrow \infty} TE(Z_1^2) \quad (\text{law of large number}) \\ &= T. \end{aligned}$$

- Q_n is random for any fixed n , but T is non-random

- Smooth functions have QV = 0:

- e.g. for $f(t) = t$, we have $\sum_{i=1}^{n-1} (t_{i+1} - t_i)^2 = n \frac{T^2}{n^2} \rightarrow 0$.

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Stochastic Integral

- Since Brownian Motions $\{W_t\}$ is non-differentiable, the **stochastic derivative** $\frac{dW_t}{dt}$ cannot be defined.
- Nevertheless, for a random function $f(t)$, the **Stochastic Integral** $\int_0^T f(t) dW_t$ can be defined as

$$\int_0^T f(t) dW_t = \lim_{\substack{n \rightarrow \infty \\ 0=t_0 < t_1 < \dots < t_n=T \\ \max_j (t_j - t_{j-1}) \rightarrow 0}} \sum_{j=0}^{n-1} f(t_j) (W_{t_{j+1}} - W_{t_j}) .$$

- **Interpretation** of stochastic integrals as **trading strategies**:
 - W_t is the **value of an asset** at time t
 - $f(t_j)$ is the **amount of investment** determined for the period $[t_j, t_{j+1})$
 - $f(t_j) (W_{t_{j+1}} - W_{t_j})$ is the payoff for the period $[t_j, t_{j+1})$
 - $\int_0^T f(t) dW_t$ is the total payoff of all investments over $[0, T]$.

Stochastic Integrals: Example

Stochastic Integral: $\int_0^T f(t) dW_t = \lim_{\substack{0=t_0 < t_1 < \dots < t_n=T \\ \max_j (t_j - t_{j-1}) \rightarrow 0}} \sum_{j=0}^{n-1} f(t_j) (W_{t_{j+1}} - W_{t_j}) .$

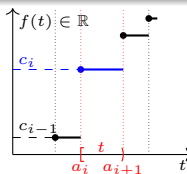
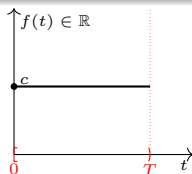
Example 4 (Stochastic Integrals)

- ① For a constant function $f(t) = c1_{[0,T]}$,

$$\int_0^T f(t) dW_t = \lim \sum_{j=0}^{n-1} f(t_j) (W_{t_{j+1}} - W_{t_j}) = \sum_{j=0}^{n-1} c (W_{t_{j+1}} - W_{t_j}) = cW_T .$$

- ② For a step function $f(t) = \sum_{i=0}^{K-1} c_i 1_{[a_i, a_{i+1})}(t)$

$$\int_0^T f(t) dW_t = \lim \sum_{j=0}^{n-1} f(t_j) (W_{t_{j+1}} - W_{t_j}) = \sum_{i=0}^{K-1} c_i (W_{a_{i+1}} - W_{a_i}) ,$$



Stochastic Integrals: Example

Stochastic Integral: $\int_0^T f(t) dW_t = \lim_{\substack{0=t_0 < t_1 < \dots < t_n=T \\ \max_j (t_j - t_{j-1}) \rightarrow 0}} \sum_{j=0}^{n-1} f(t_j) (W_{t_{j+1}} - W_{t_j}) .$

Quadratic Variation (QV): For dense $t_0, \dots, t_n$ in $[0, T]$, $Q_n = \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 \rightarrow T$.

Example 5 (Stochastic Integrals)

1 For $f(t) = W_t$,

$$\begin{aligned} \int_0^T f(t) dW_t &= \lim \sum_{j=0}^{n-1} f(t_j) (W_{t_{j+1}} - W_{t_j}) = \lim \sum_{j=0}^{n-1} W_{t_j} (W_{t_{j+1}} - W_{t_j}) \\ &= \lim \frac{1}{2} \left[\sum_{i=0}^{n-1} (W_{t_{i+1}}^2 - W_{t_i}^2) - \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 \right] \\ &\quad \left(\text{using } a(b-a) = \frac{1}{2}(b^2 - a^2) - \frac{1}{2}(b-a)^2 \right) \\ &= \frac{1}{2} W_T^2 - \lim \frac{1}{2} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 \stackrel{\text{QV}}{=} \frac{1}{2} W_T^2 - \frac{1}{2} T \end{aligned}$$

• Compare with $\int_0^T w dw = \frac{1}{2} w^2$ in ordinary calculus

Stochastic Integrals: Example

The choice of the location of integrand is important in stochastic integral:

- ① **Ito integral**: $\int_0^T f(t) dW_t = \lim_{j \rightarrow \infty} \sum_{j=0}^{n-1} f(t_j) (W_{t_{j+1}} - W_{t_j})$.
- ② **Stratonovich integral**: $\int_0^T f(t) dW_t = \lim_{j \rightarrow \infty} \sum_{j=0}^{n-1} \frac{f(t_{j+1}) + f(t_j)}{2} (W_{t_{j+1}} - W_{t_j})$
- ③ **Hänggi-Klimontovich integral**: $\int_0^T f(t) dW_t = \lim_{j \rightarrow \infty} \sum_{j=0}^{n-1} f(t_{j+1}) (W_{t_{j+1}} - W_{t_j})$

Example 6

● For $f(t) = W_t$, ① **Ito**: $\int_0^T f(t) dW_t = \frac{1}{2} W_T^2 - \frac{1}{2} T$

② **Stratonovich**: (using $(b+a)(b-a) = b^2 - a^2$)

$$\int_0^T f(t) dW_t \approx \sum_{j=0}^{n-1} \frac{W_{t_{j+1}} + W_{t_j}}{2} (W_{t_{j+1}} - W_{t_j}) = \frac{1}{2} \sum_{j=0}^{n-1} (W_{t_{j+1}}^2 - W_{t_j}^2) = \frac{1}{2} W_T^2$$

③ **Hänggi-Klimontovich**: (using $b(b-a) = \frac{1}{2}(b^2 - a^2) + \frac{1}{2}(b-a)^2$)

$$\begin{aligned} \int_0^T f(t) dW_t &\approx \sum_{j=0}^{n-1} W_{t_{j+1}} (W_{t_{j+1}} - W_{t_j}) = \frac{1}{2} \sum_{j=0}^{n-1} (W_{t_{j+1}}^2 - W_{t_j}^2) + \frac{1}{2} \sum_{j=0}^{n-1} (W_{t_{j+1}} - W_{t_j})^2 \\ &\approx \frac{1}{2} W_T^2 + \frac{1}{2} T. \end{aligned}$$

● We focus on Ito Integral due to its connection with trading strategies.

Properties of the Stochastic Integral

$$\text{Stochastic (Ito) Integral: } \int_0^T f(t) dW_t = \lim_{\substack{0=t_0 < t_1 < \dots < t_n=T \\ \max_j (t_j - t_{j-1}) \rightarrow 0}} \sum_{j=0}^{n-1} f(t_j) \underbrace{(W_{t_{j+1}} - W_{t_j})}_{\Delta_j}.$$

① **Linearity:** $\int_0^T (\alpha f(t) + \beta g(t)) dW_t = \alpha \int_0^T f(t) dW_t + \beta \int_0^T g(t) dW_t$

$$\text{LHS} \approx \sum_{j=0}^{n-1} (\alpha f(t_j) + \beta g(t_j)) \Delta_j = \alpha \sum_{j=0}^{n-1} f(t_j) \Delta_j + \beta \sum_{j=0}^{n-1} g(t_j) \Delta_j \approx \text{RHS}$$

② **Isometry:** $\mathbb{E} \left(\left[\int_0^T f(t) dW_t \right]^2 \right) = \int_0^T \mathbb{E}(f^2(t)) dt$

$$\begin{aligned} \text{LHS} &\approx \mathbb{E} \left[\left(\sum_{i=0}^{n-1} f(t_i) \Delta_i \right)^2 \right] = \mathbb{E} \left[\sum_{i=0}^{n-1} f^2(t_i) \Delta_i^2 + \sum_{i \neq j} f(t_i) f(t_j) \Delta_i \Delta_j \right] \\ &= \sum_{i=0}^{n-1} \mathbb{E} [f^2(t_i)] \mathbb{E}(\Delta_i^2) + 0 = \sum_{i=0}^{n-1} \mathbb{E}[f^2(t_i)] (t_{i+1} - t_i) \\ &= \int_0^T \mathbb{E}(f^2(t)) dt = \text{RHS} \end{aligned}$$

Properties of Stochastic Integrals

$$\text{Stochastic (Ito) Integral: } \int_0^T f(t) dW_t = \lim_{\substack{0=t_0 < t_1 < \dots < t_n=T \\ \max_j (t_j - t_{j-1}) \rightarrow 0}} \sum_{j=0}^{n-1} f(t_j) \underbrace{(W_{t_{j+1}} - W_{t_j})}_{\Delta_j}.$$

③ **Martingale Property:** $\mathbb{E} \left(\int_0^T f(t) dW_t \middle| \mathcal{F}_s \right) = \int_0^s f(t) dW_t$

- $\mathcal{F}_s$ is information up to time s .
- M_t is a martingale if $\mathbb{E}(M_t | \mathcal{F}_s) = M_s$ for $s \leq t$.

$$\begin{aligned} \text{LHS} &= \mathbb{E} \left(\sum_{j=1}^n f(t_{j-1})(W_{t_j} - W_{t_{j-1}}) \middle| \mathcal{F}_s \right) \text{ (let } s = t_k \text{ for some } k) \\ &= \sum_{j=1}^k f(t_{j-1})(W_{t_j} - W_{t_{j-1}}) + \mathbb{E} \left(\sum_{j=k+1}^n f(t_{j-1}) \overbrace{(W_{t_j} - W_{t_{j-1}})}^{\text{independent of } \mathcal{F}_s} \middle| \mathcal{F}_s \right) \\ &= \sum_{j=1}^k f(t_{j-1})(W_{t_j} - W_{t_{j-1}}) + \sum_{j=k+1}^n \mathbb{E}(f(t_{j-1}) | \mathcal{F}_s) \overbrace{E(W_{t_j} - W_{t_{j-1}})}^{=0} \\ &= \text{RHS} \end{aligned}$$

④ **Zero-mean Property:** $\mathbb{E} \left(\int_0^T f(t) dW_t \right) = 0$

(Put $s = 0$ in ③)

Properties of Stochastic Integrals

- 1 **Linearity:** $\int_0^T (\alpha f(t) + \beta g(t)) dW_t = \alpha \int_0^T f(t) dW_t + \beta \int_0^T g(t) dW_t$
- 2 **Isometry:** $\mathbb{E} \left(\left[\int_0^T f(t) dW_t \right]^2 \right) = \int_0^T \mathbb{E}(f^2(t)) dt$
- 3 **Martingale Property:** $\mathbb{E} \left(\int_0^T f(t) dW_t \middle| \mathcal{F}_s \right) = \int_0^s f(t) dW_t$

Example 7

- 1 Find $\int_0^T (T + W_t) dW_t$

Ans: LHS $\stackrel{1}{=} \int_0^T T dW_t + \int_0^T W_t dW_t = TW_T + \frac{1}{2}(W_T^2 - T)$

- 2 Find $E \left(\int_0^T \frac{W_t}{\sqrt{1-t}} dW_t \right)^2$.

Ans: LHS $\stackrel{2}{=} \int_0^T \frac{E(W_t^2)}{1-t} dt = \int_0^T \frac{t}{1-t} dt = \begin{cases} -T - \ln(1-T), & T < 1, \\ \infty, & T \geq 1. \end{cases}$

- 3 Find $\text{Var} \left(\int_0^T W_t dW_t \right)$.

Ans: LHS $\stackrel{3}{=} E \left(\int_0^T W_t dW_t \right)^2 \stackrel{2}{=} \int_0^T E(W_t^2) dt = \int_0^T t dt = \frac{1}{2}T^2$

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Itô's process

- Using stochastic integrals, an **Itô's process** $\{X_t\}$ is a model of the form

$$X_t = \int_0^t \mu(s) ds + \int_0^t \sigma(s) dW_s, \quad (1)$$

where $\mu(s)$ & $\sigma(s)$ are random functions depending on $\mathcal{F}_s$ (info up to s).

- Although X_t is not differentiable in general, to highlight the dynamics we often write (1) as

$$dX_t = \mu(t)dt + \sigma(t)dW_t.$$

- e.g. **Geometric Brownian Motion**: $S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$ & $X_t = \log \frac{S_t}{S_0}$:

$$X_t = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t = \int_0^t \left(r - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dW_s.$$

- We often consider functions $F(t, X_t)$ of X_t ,
 - e.g. discounted payoff $F(t, X_t) = e^{-rt}(X_t - K)^+$
- To discuss the dynamics $dF(t, X_t)$ in terms of dX_t , we need a **stochastic version of the chain rule**

Itô's Lemma

Lemma (Itô's Lemma)

Consider an Itô process X_t satisfying $dX_t = \mu(t)dt + \sigma(t)dW_t$, where

- W_t is a standard Brownian motion
- $\forall T > 0, \mathbb{E} \left(\int_0^T \mu^2(t) dt \right) < \infty$ & $\mathbb{E} \left(\int_0^T \sigma^2(t) dt \right) < \infty$. (so $\text{Var}(X_t)$ exists)

If $Y_t = F(t, X_t)$ for some function $F(t, x)$, then

$$dY_t = \left(\frac{\partial F}{\partial t}(t, X_t) + \frac{\partial F}{\partial x}(t, X_t)\mu(t) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(t, X_t)\sigma^2(t) \right) dt + \frac{\partial F}{\partial x}(t, X_t)\sigma(t)dW_t$$

- Taylor expanding $F(t, X_t)$ involves $dt, dW_t, dt dW_t, (dt)^2, (dW_t)^2, \dots$
 - $W_t \sim N(0, t) \Rightarrow dW_t = W_{t+dt} - W_t \sim N(0, dt) = \sqrt{dt}N(0, 1)$ (order $\sqrt{dt}$).
 - $(dW_t)^2 \sim dt(N^2(0, 1)) \approx dt$
 - $dt dW_t, (dt)^2$ and higher terms are smaller than dt , so can be omitted
 - Ito multiplication table:

$\times$	dt	dW_t
dt	0	0
dW_t	0	dt

Itô's Lemma: The Proof

Itô process: $dX_t = \mu(t)dt + \sigma(t)dW_t$

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

• dW_t is of order $\sqrt{dt}$, $(dW_t)^2 = dt$, $(dt)^2, dt dW_t, (dt)^3, dt(dW_t)^2, \dots$ negligible

Proof.

Consider the Taylor's expansion of F up to terms of order in dt :

$$\begin{aligned} dF &= F(t + dt, X_{t+dt}) - F(t, X_t) \\ &= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} dX_t + \frac{1}{2} \left(\frac{\partial^2 F}{\partial t^2} (dt)^2 + 2 \frac{\partial^2 F}{\partial t \partial x} (dt dX_t) + \frac{\partial^2 F}{\partial x^2} (dX_t)^2 \right) + \dots \\ &= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} (\mu(t)dt + \sigma(t)dW_t) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} (\mu(t)dt + \sigma(t)dW_t)^2 + \dots \\ &\approx \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu(t) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2(t) \right) dt + \frac{\partial F}{\partial x} \sigma(t) dW_t \end{aligned}$$



Itô's Lemma: Examples

Itô process: $dX_t = \mu(t)dt + \sigma(t)dW_t$

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

Example 8

Find $d \log S_t$, where $dS_t = \mu S_t dt + \sigma S_t dW_t$.

① Identify variables: $F(t, S_t) = \log S_t$, $X_t = S_t$, $\mu(t) = \mu S_t$, $\sigma(t) = \sigma S_t$

② Find derivatives: $\frac{\partial F}{\partial t} = 0$, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial S_t} = \frac{1}{S_t}$, $\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial S_t^2} = -\frac{1}{S_t^2}$

③ Apply Itô's:

$$\begin{aligned} d(\log S_t) &= \left(0 + \frac{1}{S_t} (\mu S_t) + \left(-\frac{1}{S_t^2} \right) (\sigma^2 S_t^2) \right) dt + \frac{1}{S_t} (\sigma S_t) dW_t \\ &= \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t \end{aligned}$$

• Ordinary calculus: $d \log x = \frac{dx}{x}$

• Stochastic calculus: $d \log S_t = \mu dt + \sigma dW_t - \frac{1}{2} \sigma^2 dt = \frac{dS_t}{S_t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} (dS_t)^2$

Itô's Lemma: Examples

Itô process: $dX_t = \mu(t)dt + \sigma(t)dW_t$

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

Example 9

Evaluate $d \log X_t$, where $X_0 = 1$, $dX_t = \frac{1}{2} X_t dt + X_t dW_t$

- 1 Identify variables: $F(t, X_t) = \log X_t$, $\mu(t) = \frac{X_t}{2}$, $\sigma(t) = X_t$
- 2 Find derivatives: $\frac{\partial F}{\partial t} = 0$, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial X_t} = \frac{1}{X_t}$, $\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial X_t^2} = -\frac{1}{X_t^2}$
- 3 Apply Itô's:

$$d(\log X_t) = \left(0 + \frac{1}{X_t} \frac{X_t}{2} - \frac{1}{2} \frac{1}{X_t^2} X_t^2 \right) dt + \frac{1}{X_t} X_t dW_t = dW_t$$

- 4 Conclusion: $\log X_t = W_t$, or $X_t = e^{W_t}$

Itô's Lemma: Examples

Itô process: $dX_t = \mu(t)dt + \sigma(t)dW_t$

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

Example 10

Evaluate dW_t^3

- 1 Identify variables: $F = W_t^3$, $X_t = W_t \Rightarrow dX_t = dW_t$, $\mu(t) = 0$, $\sigma(t) = 1$
- 2 Find derivatives: $\frac{\partial F}{\partial t} = 0$, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial W_t} = 3W_t^2$, $\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial W_t^2} = 6W_t$
- 3 Apply Itô's: $d(W_t^3) = 3W_t dt + 3W_t^2 dW_t$

Itô's Lemma: Examples

Itô process: $dX_t = \mu(t)dt + \sigma(t)dW_t$

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

Example 11

Evaluate de^{X_t} , where $X_0 = 1$, $dX_t = \frac{1}{2}dt + dW_t$

- 1 Identify variables: $F(t, X_t) = e^{X_t}$, $\mu(t) = \frac{1}{2}$, $\sigma(t) = 1$
- 2 Find derivatives: $\frac{\partial F}{\partial t} = 0$, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial X_t} = e^{X_t}$, $\frac{\partial F^2}{\partial x^2} = \frac{\partial F^2}{\partial X_t^2} = e^{X_t}$
- 3 Apply Itô's:

$$de^{X_t} = \left(0 + e^{X_t} \frac{1}{2} + \frac{1}{2} e^{X_t} \right) dt + e^{X_t} dW_t = e^{X_t} (dt + dW_t)$$

Itô's Lemma: Examples

Itô process: $dX_t = \mu(t)dt + \sigma(t)dW_t$

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

Example 12

Find $\int_0^T W_t dW_t$ by Itô lemma

- ❶ Guess: $\frac{1}{2} W_T^2$
- ❷ Identify variables: $F = \frac{1}{2} W_t^2$, $X_t = W_t \Rightarrow dX_t = dW_t$, $\mu(t) = 0$, $\sigma(t) = 1$
- ❸ Find derivatives: $\frac{\partial F}{\partial t} = 0$, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial W_t} = W_t$, $\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial W_t^2} = 1$
- ❹ Apply Itô's: $d(\frac{1}{2} W_t^2) = \frac{1}{2} dt + W_t dW_t$
 $\Rightarrow \frac{1}{2} W_T^2 = \frac{1}{2} T + \int_0^T W_t dW_t$
- ❺ Conclusion: $\int_0^T W_t dW_t = \frac{1}{2} W_T^2 - \frac{T}{2}$

Itô's Lemma: Examples

Itô process: $dX_t = \mu(t)dt + \sigma(t)dW_t$

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

Example 13

Find $\int_0^T t dW_t$ by Itô lemma

- ➊ **Guess:** $TW_T - \int_0^T W_t dt$ (integration by part In ordinary calculus)
 - ➋ **Identify variables:** $F = tW_t$, $X_t = W_t \Rightarrow dX_t = dW_t$, $\mu(t) = 0$, $\sigma(t) = 1$
 - ➌ **Find derivatives:** $\frac{\partial F}{\partial t} = W_t$, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial W_t} = t$, $\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial W_t^2} = 0$
 - ➍ **Apply Itô's:** $d(tW_t) = W_t dt + t dW_t$
 - ➎ **Conclusion:** $\int_0^T t dW_t = TW_T - \int_0^T W_t dt$
- The results of ordinary and stochastic calculus agree since $\frac{\partial^2 F}{\partial x^2} = 0$.

Itô's Lemma: Examples

Itô process: $dX_t = \mu(t)dt + \sigma(t)dW_t$

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

Example 14

Using Itô lemma on $F(t, X_t) = e^{\alpha t} X_t$, solve the SDE

$$dX_t = -\alpha X_t dt + \sigma dW_t. \quad (2)$$

- 1 Identify variables $F(t, X_t) = e^{\alpha t} X_t$, $\mu(t) = -\alpha X_t$, $\sigma(t) = \sigma$
- 2 Find derivatives: $\frac{\partial F}{\partial t} = \alpha e^{\alpha t} X_t$, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial X_t} = e^{\alpha t}$, $\frac{\partial F^2}{\partial x^2} = \frac{\partial F^2}{\partial X_t^2} = 0$
- 3 Apply Itô's:

$$d(e^{\alpha t} X_t) = \left(\alpha e^{\alpha t} X_t + e^{\alpha t} (-\alpha X_t) + 0 \right) dt + \sigma e^{\alpha t} dW_t = \sigma e^{\alpha t} dW_t$$

$$\Rightarrow X_T = X_0 e^{-\alpha T} + \sigma \int_0^T e^{-\alpha(T-t)} dW_t.$$

- Process (2) is called **Ornstein-Uhlenbeck (OU) Process**, and is applied in modeling bond prices.

Agenda

- 1 Functional Central Limit Theorem
- 2 Brownian Motion and Wiener Process
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- 6 Black-Scholes PDE

Product Rule

Theorem 3 (Product Rule)

If $dX_t = \mu_x(t)dt + \sigma_x(t)dW_t$ and $dY_t = \mu_y(t)dt + \sigma_y(t)dW_t$, then

$$\begin{aligned}dX_t Y_t &= X_t dY_t + Y_t dX_t + dX_t dY_t \\ &= X_t dY_t + Y_t dX_t + \sigma_x(t)\sigma_y(t) dt.\end{aligned}$$

Proof.

$$\begin{aligned}dX_t Y_t &= X_{t+dt} Y_{t+dt} - X_t Y_t \\ &= X_t (Y_{t+dt} - Y_t) + Y_t (X_{t+dt} - X_t) + (Y_{t+dt} - Y_t)(X_{t+dt} - X_t) \\ &= X_t dY_t + Y_t dX_t + dX_t dY_t \\ &= X_t dY_t + Y_t dX_t + (\mu_x(t)dt + \sigma_x(t)dW_t)(\mu_y(t)dt + \sigma_y(t)dW_t) \\ &= X_t dY_t + Y_t dX_t + \sigma_x(t)\sigma_y(t) dt,\end{aligned}$$

by keeping only terms of order up to dt



Product Rule: Example

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

Product Rule: $dX_t Y_t = X_t dY_t + Y_t dX_t + dX_t dY_t$

Example 15 (Product Rule)

- Find dX_t^3 , where $dX_t = \mu X_t dt + \sigma X_t dW_t$.
- Method 1:** Using Itô's lemma with $F(t, X_t) = X_t^3$,

$$dX_t^3 = 3X_t^2 dX_t + 3X_t (dX_t)^2.$$

- Method 2:** Using product rule with $X_t = X_t$ and $Y_t = X_t^2$,

$$\begin{aligned} dX_t^3 &= d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t \\ &= X_t^2 dX_t + X_t (2X_t dX_t + (dX_t)^2) + dX_t (2X_t dX_t + (dX_t)^2) \\ &= 3X_t^2 dX_t + 3X_t (dX_t)^2 + (dX_t)^3 \\ &= 3X_t^2 dX_t + 3X_t (dX_t)^2 \quad ((dX_t)^3 \text{ is negligible}) \end{aligned}$$

Integration By Parts

Product Rule: $dX_t Y_t = X_t dY_t + Y_t dX_t + dX_t dY_t$,

By directly integrating the product rule, we have the following result:

Theorem 4 (Integration by Parts)

$$\int_a^b X_t dY_t = [X_t Y_t]_a^b - \int_a^b Y_t dX_t - \int_a^b dX_t dY_t$$

Example 16 (Integration by Parts)

Find $\int_0^T W_t dW_t$.

① Let $X_t = W_t$, $Y_t = W_t$. Applying integration by parts gives

$$\begin{aligned}\int_0^T W_t dW_t &= [W_t \times W_t]_0^T - \int_0^T W_t dW_t - \int_0^T dW_t dW_t \\ &= W_T^2 - \int_0^T W_t dW_t - \int_0^T dt \\ \Rightarrow \int_0^T W_t dW_t &= \frac{1}{2} W_T^2 - \frac{1}{2} T.\end{aligned}$$

Integration By Parts

Product Rule: $dX_t Y_t = X_t dY_t + Y_t dX_t + dX_t dY_t$

Integration by Parts: $\int_a^b X_t dY_t = [X_t Y_t]_a^b - \int_a^b Y_t dX_t - \int_a^b dX_t dY_t$

Example 17 (Integration by Parts)

Find $\int_0^T t W_t dW_t$.

① Let $X_t = tW_t$, $Y_t = W_t$. By Product rule, $dX_t = d(tW_t) = W_t dt + t dW_t$

② Applying integration by parts,

$$\begin{aligned}\int_0^T t W_t dW_t &= [tW_t \times W_t]_0^T - \int_0^T W_t d(tW_t) - \int_0^T d(tW_t) dW_t \\&= TW_T^2 - \int_0^T W_t (W_t dt + t dW_t) - \int_0^T (W_t dt + t dW_t) dW_t \\&= TW_T^2 - \int_0^T W_t^2 dt - \int_0^T t W_t dW_t - \int_0^T t dt \quad (dt dW_t \text{ neglected}) \\ \Rightarrow \int_0^T t W_t dW_t &= \frac{1}{2} TW_T^2 - \frac{1}{2} \int_0^T W_t^2 dt - \frac{T^2}{4}.\end{aligned}$$

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- 1 Functional Central Limit Theorem
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Black-Scholes Partial Differential Equation

Theorem 5 (Black-Scholes PDE [SDE involves dW_t ; PDE does not])

Suppose that the market has the following asset price dynamics:

- **Security (Stock):** $dS_t = \mu S_t dt + \sigma S_t dW_t$,
- **Risk-free asset (Bond):** $dB_t = r B_t dt$.

The price of any derivative of S_t , denoted as $f(t, S_t)$, satisfies, $\forall t \geq 0$,

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} r S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2 = r f.$$

- The proof relies on mimicking payoff f by a **replicating portfolio**

$$V_t = V(t, S_t) = \phi_t S_t + \theta_t B_t,$$

where ϕ_t, θ_t are units of holdings of stocks and bonds, respectively.

- Properties of the replicating portfolio;

① **Replicating:** $f(t, S_t) = V(t, S_t)$ for all $t \geq 0$

② **Self-financing:** $dV_t = \phi_t dS_t + \theta_t dB_t$

- Change in V_t is only from changes in S_t & $B_t \Rightarrow$ no new investment

Black-Scholes Partial Differential Equation

Itô process: $dX_t = \mu(t)dt + \sigma(t)dW_t$

Itô's Lemma: $dY_t = dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

Stock: $dS_t = \mu S_t dt + \sigma S_t dW_t$, **Bond:** $dB_t = r B_t dt$, **Derivative:** $f = f(t, S_t)$

Black-Scholes PDE: $\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} r S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 = r f$, $dS_t = \mu S_t dt + \sigma S_t dW_t$

Proof of Black-Scholes PDE.

- ❶ Identify variables: $X_t = S_t$, $\mu(t) = \mu S_t$, $\sigma(t) = \sigma S_t$, $F(t, X_t) = f(t, X_t)$
- ❷ Apply Itô's: $df = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} \mu S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2 \right] dt + \left[\frac{\partial f}{\partial S_t} \sigma S_t \right] dW_t$
- ❸ Replicating portfolio: $dV_t = \phi_t dS_t + \theta_t dB_t$, (weights: ϕ_t stocks, θ_t bonds)
 $= (\phi_t \mu S_t + \theta_t r B_t) dt + \phi_t \sigma S_t dW_t$
- ❹ No arbitrage $\Rightarrow df = dV_t \Rightarrow$ ❶ $\phi_t = \frac{\partial f}{\partial S_t}$ & ❷ $\theta_t r B_t = \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2$;
- ❺ No arbitrage $\Rightarrow f = V \Rightarrow$ ❸ $f = \phi_t S_t + \theta_t B_t$;
- ❻ Combining ❶ and ❸ $\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} r S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2 = r f$



Black-Scholes PDE: Example

Black-Scholes Model: $dS_t = \mu S_t dt + \sigma S_t dW_t$

Black-Scholes PDE: $\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} r S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2 = r f$ (derivative $f(t, S_t)$)

Example 18 (Black-Scholes PDE)

① **Stock** $f(t, S_t) = S_t$:

- $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial S_t} = 1$, and $\frac{\partial^2 f}{\partial S_t^2} = 0$
- BS-PDE reduces to $r S_t = r S_t$

② **Bond** $f(t, S_t) = e^{rt}$:

- $\frac{\partial f}{\partial t} = r e^{rt}$, $\frac{\partial f}{\partial S_t} = 0$, and $\frac{\partial^2 f}{\partial S_t^2} = 0$
- BS-PDE reduces to $r e^{rt} = r e^{rt}$

③ **Forward** $f(t, S_t) = S_t - K e^{-r(T-t)}$:

- $\frac{\partial f}{\partial t} = -r K e^{-r(T-t)}$, $\frac{\partial f}{\partial S_t} = 1$, $\frac{\partial^2 f}{\partial S_t^2} = 0$
- BS-PDE reduces to $-r K e^{-r(T-t)} + r S_t = r f = r (S_t - K e^{-r(T-t)})$

Black-Scholes PDE: Discussions

Black-Scholes Model: $dS_t = \mu S_t dt + \sigma S_t dW_t$

Black-Scholes PDE: $\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} r S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2 = r f$ (derivative $f(t, S_t)$)

- BS-PDE provides a way to price financial derivatives, i.e., find $f(t, S_t)$, by analytically or numerically solving a PDE
- The desired price $f(t, S_t)$ is a solution to the BS-PDE under suitable initial/boundary conditions
 - **Stock:** $f(0, S_0) = S_0, f(T, S_T) = S_T$
 - **Bond:** $f(0, S_0) = 1, f(T, S_T) = e^{rT}$
 - **European Call:** $f(t, 0) = 0$ & $f(T, S_T) = (S_T - K)^+$
 - **American Call:** additionally, $f(t, S_t) \geq (S_t - K)^+$
 - **European Put:** $f(t, \infty) = 0$ & $f(T, S_T) = (K - S_T)^+$
 - ...
- see RMSC 4007 for details

Summary

- **Brownian Motion/Wiener Process $\{W_t\}$:**

- **Stationary increment:** For $s < t$, $W_t - W_s \sim N(0, t - s)$
- **Independent increment:** For $0 \leq t_1 < t_2 \leq t_3 < t_4$, $W_{t_4} - W_{t_3} \perp W_{t_2} - W_{t_1}$
- **Starts at 0:** $\mathbb{P}(\{W_0 = 0\}) = 1$.

- **Stochastic Integral:**

$$\int_0^T f(t) dW_t = \lim_{\substack{n \rightarrow \infty \\ 0=t_0 < t_1 < \dots < t_n=T \\ \max_j (t_j - t_{j-1}) \rightarrow 0}} \sum_{j=0}^{n-1} f(t_j) (W_{t_{j+1}} - W_{t_j})$$

- **Linearity:** $\int_0^T (\alpha f(t) + \beta g(t)) dW_t = \alpha \int_0^T f(t) dW_t + \beta \int_0^T g(t) dW_t$
- **Isometry:** $\mathbb{E} \left(\left[\int_0^T f(t) dW_t \right]^2 \right) = \int_0^T \mathbb{E}(f^2(t)) dt$
- **Martingale:** $\mathbb{E} \left(\int_0^T f(t) dW_t \middle| \mathcal{F}_s \right) = \int_0^s f(t) dW_t$ (zero-mean when $s = 0$)

- **Itô process:** $dX_t = \mu(t)dt + \sigma(t)dW_t$

- **Itô's Lemma:** $dF(t, X_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 \right) dt + \frac{\partial F}{\partial x} \sigma dW_t$

- **Black-Scholes Model:** $dS_t = \mu S_t dt + \sigma S_t dW_t$

- **Black-Scholes PDE:** $\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} r S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2 = r f$ (derivative $f(t, S_t)$)