

香 港 中 文 大 學
The Chinese University of Hong Kong

Course Examinations, 1st Term, 2022-23 年度下學期科目考試

Course Code & Title 科目編號及名稱: RMSC4003: Statistical Modeling in Financial Markets

Time allowed 時間: 2 hours 小時 0 minutes 分鐘

Student I.D. No. 學生編號: Seat No. 座號:

Answer all the questions

1) (20 marks) An excel file “FamaFrench.csv” contains the following data set.

Date	r	Return1	Return2	...	Return8	RM	SMB	HML	RMW	CMA
2010-01-31	0.004	0.03	-0.023	...	0.15	0.21	-0.23	0.032	0.54	-0.64
2010-02-28	...			...						
...	...			...						
2021-12-31	...			...						

Here, r is the risk-free rate, Return1 to Return8 are the returns of eight stocks (Stock 1 to Stock 8), RM is the return of a proxy of the market portfolio, SMB to CMA are the variables used in the Fama-French five factor model.

- a) (13 marks) Write down the R codes for
- i) (1 mark) Read the file in R.
 - ii) (2 marks) computing the **excess return** of the market portfolio return, and store the results in a vector ERM.
 - iii) (2 marks) merging ERM, SMB to CMA into a design matrix Z.
 - iv) (5 marks) conducting time series regression and store the results in a matrix b. Give interpretation for b.
 - v) (3 marks) conducting cross sectional regression and store the results in a matrix temp.
- b) (3 marks) Explain what arbitrage pricing theory states about the factor model

$$\text{Return}_i = a_i + \mathbf{b}_i^T \mathbf{F} + e_i, \tag{1}$$

where $i = 1, \dots, 8$, Return_i is the return of Stock i , $\mathbf{F}$ is the collection of factors in iii).

- c) (4 marks) Write down the R codes to compute the factor risk premium λ_0 and $\boldsymbol{\lambda}$ for the factor model in b).

- 2) (10 marks) Let $q = \frac{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{r\delta}}{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{\mu\delta - \sigma\sqrt{\delta}}}$.
- a) (3 marks) Find the limit of q as $\delta \rightarrow 0$.
 - b) (3 marks) Find the limit of $\frac{2q-1}{\sqrt{\delta}}$ as $\delta \rightarrow 0$.
 - c) (4 marks) Find the limit of $\frac{2q-1 - \frac{1}{\sigma}(\mu-r+\frac{1}{2}\sigma^2)\sqrt{\delta}}{\delta}$ as $\delta \rightarrow 0$.
- 3) (5 marks) Let m be a non-negative integer. Find an alternative expression of $\int_0^T t^m dW_t$ which involves only a Riemann integral but not a stochastic integral.
- 4) (15 marks) Let the stock price S_t follow a Black-Scholes model with risk-free rate $r = 0.01$ and volatility $\sigma = 0.2$. At time 0, $S_0 = 100$. Consider an option which is worth $(\log S_{10} - 2)^+$ at time 10. Price the option.

5) (20 marks) Consider a financial market consisting of three assets with the following structure:

- Initially, the values of the assets are $\mathbf{S}^0 = (S_1^0, S_2^0, S_3^0)' = (1, 1, 2)'$.
- At the maturity time T , there are four scenarios. The possible values are collected as

$$\mathbf{S} = \begin{pmatrix} 1.3 & 0.8 & 0 \\ 0.9 & 1.1 & 0 \\ 0 & 0 & 1.2 \\ 0 & 0 & 2.4 \end{pmatrix},$$

where the (i, j) entry of $\mathbf{S}$ is the value of Stock j in the i -th scenario.

- (3 marks) Is the market complete? Explain.
- (4 marks) Recall that a state price vector is a vector $\boldsymbol{\psi}$ which satisfies $(\mathbf{S}^0)' = \boldsymbol{\psi}'\mathbf{S}$. Find a state price vector for the market. Is the state price vector unique?
- (4 marks) Is there any risk-free asset in the market? Explain.
- (3 marks) Is there any arbitrage opportunity in the market? If yes, prove it. Otherwise, give an example of arbitrage portfolio.
- (3 marks) Find a replicating portfolio of payoffs taking values $(1, 2, 4, 8)$ in the four scenarios.
- (3 marks) Price a derivative which has a payoffs $(1, 2, 4, 8)$ in the four scenarios.

6) (10 marks) Let B_t be a standard Brownian motion.

- (5 marks) By using polar coordinate, show that

$$\int_0^\infty \int_{-x}^\infty \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dy dx = \frac{3}{8}.$$

- (5 marks) By using *a*), or otherwise, find $\mathbb{P}(B_1 > 0, B_2 > 0)$.

7) (10 marks) Let $\{B_t\}$ be a standard Brownian motion. Show that $X_t = B_t^2 - t$ satisfies the martingale property that $E(X_t | \mathcal{F}_s) = X_s$ for $s < t$.

8) (10 marks) Let $\{B_t\}$ be a standard Brownian motion. Solve the stochastic differential equation

$$dX_t = -\frac{1}{2}X_t dt + \sqrt{1 - X_t^2} dB_t,$$

with $X_0 = 0$. Hint: try a trigonometry function.

— END OF THE PAPER —