

SOLUTION TO FINAL EXAM

i) a) i) `data = read.csv("FamaFrench.csv")`

ii) `ERM = data$ReM - data$r` OR
`ERM = data[,11]-data[,2]`

iii) `Z = matrix(c(ERM,data$SMB, data$HML, data$RMW, data$CMA),ncol=5)` OR
`Z = matrix(c(ERM,data[,12],data[,13],data[,14],data[,15]),ncol=5)`

iv) `b = matrix(0,8,5)`
`for (i in 1:8){`
`b[i,] = lm(data[,2+i]~Z)$coef[-1]`
`}`
 b_i denotes the factor loadings/sensitivities for each asset.

v) `T = dim(data)[1]`
`res = matrix(0,T,8)`
`for (t in 1:T){`
`temp = as.numeric(data[t,3:10])`
`res[t,] = lm(temp~b)$coef`
`}`

b) Arbitrage Pricing Theory states that under the K -factor model $R_i = a_i + \mathbf{b}_i^T \mathbf{F} + e_i$, $i = 1, \dots, n$, there exists constant λ_0 , $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_K)^T$ (factor risk premiums) such that

$$\mu_i = \lambda_0 + \mathbf{b}_i^T \boldsymbol{\lambda}$$

c) `lambda0 = mean(res[,1])`
`lambda = apply(res[,2:6],2,mean)`

2) Denote $O(\delta^k)$ as a term that is of size comparable or smaller than δ^k . Recall $e^x = 1 + x + \frac{x^2}{2} + \frac{1}{3!}x^3 \dots$

$$\begin{aligned}
q &= \frac{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{r\delta}}{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{\mu\delta - \sigma\sqrt{\delta}}} \\
&= \frac{\left[1 + (\mu\delta + \sigma\sqrt{\delta}) + \frac{1}{2}(\mu\delta + \sigma\sqrt{\delta})^2 + \frac{1}{6}(\mu\delta + \sigma\sqrt{\delta})^3 + O(\delta^2)\right] - [1 + r\delta + O(\delta^2)]}{\left[1 + (\mu\delta + \sigma\sqrt{\delta}) + \frac{1}{2}(\mu\delta + \sigma\sqrt{\delta})^2 + \frac{1}{6}(\mu\delta + \sigma\sqrt{\delta})^3\right] - \left[1 + (\mu\delta - \sigma\sqrt{\delta}) + \frac{1}{2}(\mu\delta - \sigma\sqrt{\delta})^2 + \frac{1}{6}(\mu\delta - \sigma\sqrt{\delta})^3\right] + O(\delta^{5/2})} \\
&= \frac{\sigma\sqrt{\delta} + \left(\mu - r + \frac{1}{2}\sigma^2\right)\delta + \left(\mu\sigma + \frac{1}{6}\sigma^3\right)\delta^{\frac{3}{2}} + O(\delta^2)}{2\sigma\sqrt{\delta} + \left(2\mu\sigma + \frac{1}{3}\sigma^3\right)\delta^{\frac{3}{2}} + O(\delta^{\frac{5}{2}})} \\
&= \frac{\sigma\sqrt{\delta} + \left(\mu - r + \frac{1}{2}\sigma^2\right)\delta + \left(\mu\sigma + \frac{1}{6}\sigma^3\right)\delta^{\frac{3}{2}} + O(\delta^2)}{2\sigma\sqrt{\delta}} \frac{1}{1 + \left(\mu + \frac{1}{6}\sigma^2\right)\delta + O(\delta^2)} \\
&= \left(\frac{1}{2} + \frac{(\mu - r + \frac{1}{2}\sigma^2)\sqrt{\delta}}{2\sigma} + \frac{(\mu + \frac{1}{6}\sigma^2)\delta}{2} + O(\delta^{\frac{3}{2}})\right) \left(1 - \left[\left(\mu + \frac{1}{6}\sigma^2\right)\delta + O(\delta^2)\right] + \left[\left(\mu + \frac{1}{6}\sigma^2\right)\delta + O(\delta^2)\right]^2 - \dots\right) \\
&\quad \left(\text{recall } \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots\right) \\
&= \left(\frac{1}{2} + \frac{(\mu - r + \frac{1}{2}\sigma^2)\sqrt{\delta}}{2\sigma} + \frac{(\mu + \frac{1}{6}\sigma^2)\delta}{2} - \frac{1}{2}\left(\mu + \frac{1}{6}\sigma^2\right)\delta + O(\delta^{\frac{3}{2}})\right) \\
&= \frac{1}{2} + \frac{(\mu - r + \frac{1}{2}\sigma^2)\sqrt{\delta}}{2\sigma} + O(\delta^{\frac{3}{2}})
\end{aligned}$$

Therefore,

a) As $\delta \rightarrow 0$, $q \rightarrow \frac{1}{2}$.

b)

$$\begin{aligned}
\frac{2q-1}{\sqrt{\delta}} &= \frac{2\left(\frac{1}{2} + \frac{\sqrt{\delta}(\mu - r + \frac{1}{2}\sigma^2)}{2\sigma} + O(\delta^{\frac{3}{2}})\right) - 1}{\sqrt{\delta}} \\
&= \frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma} + O(\delta)
\end{aligned}$$

As $\delta \rightarrow 0$, $q \rightarrow \frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}$.

c)

$$\frac{2q-1 - \frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}\sqrt{\delta}}{\delta} = O(\delta^{\frac{1}{2}})$$

As $\delta \rightarrow 0$, the term goes to 0.

Actually can go further and keep the δ^2 term...Recall $e^x = 1 + x + \frac{x^2}{2} + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \dots$

$$\begin{aligned}
q &= \frac{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{r\delta}}{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{\mu\delta - \sigma\sqrt{\delta}}} \\
&= \frac{\left[1 + (\mu\delta + \sigma\sqrt{\delta}) + \frac{1}{2}(\mu\delta + \sigma\sqrt{\delta})^2 + \frac{1}{6}(\mu\delta + \sigma\sqrt{\delta})^3 + \frac{1}{24}(\mu\delta + \sigma\sqrt{\delta})^4 + O(\delta^{\frac{5}{2}})\right] - \left[1 + r\delta + \frac{1}{2}r^2\delta^2 + O(\delta^3)\right]}{\left[1 + (\mu\delta + \sigma\sqrt{\delta}) + \frac{1}{2}(\mu\delta + \sigma\sqrt{\delta})^2 + \frac{1}{6}(\mu\delta + \sigma\sqrt{\delta})^3\right] - \left[1 + (\mu\delta - \sigma\sqrt{\delta}) + \frac{1}{2}(\mu\delta - \sigma\sqrt{\delta})^2 + \frac{1}{6}(\mu\delta - \sigma\sqrt{\delta})^3\right] + O(\delta^{\frac{5}{2}})} \\
&= \frac{\sigma\sqrt{\delta} + \left(\mu - r + \frac{1}{2}\sigma^2\right)\delta + \left(\mu\sigma + \frac{1}{6}\sigma^3\right)\delta^{\frac{3}{2}} + \left(\frac{1}{2}(\mu^2 - r^2 + \mu\sigma^2) + \frac{1}{24}\sigma^4\right)\delta^2 + O(\delta^{\frac{5}{2}})}{2\sigma\sqrt{\delta} + \left(2\mu\sigma + \frac{1}{3}\sigma^3\right)\delta^{\frac{3}{2}} + O(\delta^{\frac{5}{2}})} \\
&= \frac{\sigma\sqrt{\delta} + \left(\mu - r + \frac{1}{2}\sigma^2\right)\delta + \left(\mu\sigma + \frac{1}{6}\sigma^3\right)\delta^{\frac{3}{2}} + \left(\frac{1}{2}(\mu^2 - r^2 + \mu\sigma^2) + \frac{1}{24}\sigma^4\right)\delta^2 + O(\delta^{\frac{5}{2}})}{2\sigma\sqrt{\delta}} \frac{1}{1 + \left(\mu + \frac{1}{6}\sigma^2\right)\delta + O(\delta^2)} \\
&= \left(\frac{1}{2} + \frac{(\mu - r + \frac{1}{2}\sigma^2)\sqrt{\delta}}{2\sigma} + \frac{(\mu + \frac{1}{6}\sigma^2)\delta}{2} + \frac{\mu^2 - r^2 + \mu\sigma^2 + \frac{1}{12}\sigma^4}{4\sigma}\delta^{\frac{3}{2}} + O(\delta^2)\right) \times \\
&\quad \left(1 - \left[\left(\mu + \frac{1}{6}\sigma^2\right)\delta + O(\delta^2)\right] + \left[\left(\mu + \frac{1}{6}\sigma^2\right)\delta + O(\delta^2)\right]^2 - \dots\right) \\
&= \frac{1}{2} + \frac{(\mu - r + \frac{1}{2}\sigma^2)\sqrt{\delta}}{2\sigma} + \frac{(\mu + \frac{1}{6}\sigma^2)\delta}{2} + \frac{\mu^2 - r^2 + \mu\sigma^2 + \frac{1}{12}\sigma^4}{4\sigma}\delta^{\frac{3}{2}} - \frac{(\mu + \frac{1}{6}\sigma^2)\delta}{2} - \frac{(\mu - r + \frac{1}{2}\sigma^2)(\mu + \frac{1}{6}\sigma^2)}{2\sigma}\delta^{\frac{3}{2}} + O(\delta^2) \\
&= \frac{1}{2} + \frac{(\mu - r + \frac{1}{2}\sigma^2)\sqrt{\delta}}{2\sigma} - \frac{(\mu - r)^2 + \frac{1}{3}(\mu - r + \frac{1}{4}\sigma^2)\sigma^2}{4\sigma}\delta^{\frac{3}{2}} + O(\delta^2).
\end{aligned}$$

Therefore,

$$\frac{2q - 1 - \frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}\sqrt{\delta}}{\delta^{\frac{3}{2}}} = -\frac{(\mu - r)^2 + \frac{1}{3}(\mu - r + \frac{1}{4}\sigma^2)\sigma^2}{2\sigma} + O(\delta^{\frac{1}{2}})$$

As $\delta \rightarrow 0$, the term goes to $-\frac{(\mu - r)^2 + \frac{1}{3}(\mu - r + \frac{1}{4}\sigma^2)\sigma^2}{2\sigma}$.

The following codes verifies that the calculation is correct:

```

try=function(mu,sigma,r,delta){
num=exp(mu*delta+sigma*sqrt(delta))-exp(r*delta)
den=exp(mu*delta+sigma*sqrt(delta))-exp(mu*delta-sigma*sqrt(delta))
q=num/den
q1=(2*q-1)/sqrt(delta)
q2=(2*q-1-(mu-r+0.5*sigma^2)*sqrt(delta)/sigma)/(delta^1.5)
print("Computed Limit")
print(c(1/2, (mu-r+0.5*sigma^2)/sigma, -((mu-r)^2 +1/3*(mu-r+0.25*sigma^2)*sigma^2)/2/sigma))
print("Empirical ratio")
return(c(q,q1,q2))
}

try(mu=0.2,sigma=0.3,r=0.05,delta=1)
try(mu=0.2,sigma=0.3,r=0.05,delta=0.1)
try(mu=0.2,sigma=0.3,r=0.05,delta=0.01)
try(mu=0.2,sigma=0.3,r=0.05,delta=0.001)
try(mu=0.2,sigma=0.3,r=0.05,delta=0.0001)

```

3) By Ito's Lemma,

$$\begin{aligned}
 d(t^m W_t) &= t^m dW_t + mt^{m-1} W_t dt \\
 T^m W_T &= \int_0^T t^m dW_t + \int_0^T mt^{m-1} W_t dt \\
 \underbrace{\int_0^T t^m dW_t}_{\text{Stochastic integral}} &= T^m W_T - \underbrace{\int_0^T mt^{m-1} W_t dt}_{\text{Riemann integral}}
 \end{aligned}$$

4) Under Black-Scholes model, $dS_t = rS_t dt + \sigma S_t dW_t$ under the risk neutral measure. Apply Ito's Lemma with $F(t, x) = \log(x)$, we have

$$\begin{aligned}
 d \log S_t &= \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t \\
 \log S_T &= \log S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma dW_T
 \end{aligned}$$

By Fundamental Theorem of Asset Pricing,

$$\begin{aligned}
 v_0 &= \mathbb{E} \left\{ e^{-rT} (\log S_T - 2)^+ \right\} \\
 &= \mathbb{E} \left\{ e^{-rT} (\log S_0 + (r - \frac{\sigma^2}{2})T + \sigma dW_T - 2)^+ \right\} \\
 &= e^{-rT} \int_{-d\sqrt{T}}^{\infty} \left\{ \log(S_0) + (r - \frac{\sigma^2}{2})T + \sigma x - 2 \right\} \phi(x, 0, T) dx & (d = \frac{\log(S_0) + (r - \frac{\sigma^2}{2})T - 2}{\sigma\sqrt{T}}) \\
 &= e^{-rT} \int_{-d}^{\infty} \left\{ \log(S_0) + (r - \frac{\sigma^2}{2})T + \sigma u\sqrt{T} - 2 \right\} \phi(u) du & (u = \frac{x}{\sqrt{T}}) \\
 &= e^{-rT} \left\{ (\log(S_0) + (r - \frac{\sigma^2}{2})T - 2) \Phi(d) + e^{-rT} \int_{-d}^{\infty} \sigma u\sqrt{T} \phi(u) du \right\} \\
 &= e^{-rT} \left\{ (\log(S_0) + (r - \frac{\sigma^2}{2})T - 2) \Phi(d) + \frac{e^{-rT} \sigma \sqrt{T}}{\sqrt{2\pi}} \left(e^{-\frac{d^2}{2}} \right) \right\} \\
 &= 2.267 \Phi(d) + 8.945 \times 10^{-5}
 \end{aligned}$$

Let $r = 0.01, \sigma = 0.2, T = 10, S_0 = 100$, we have $d = 3.961$.

$$\implies v_0 = 2.267.$$

- 5) a) The market is not complete as payoffs of the form $\mathbf{F} = (a, b, c, d)'$ with $2c \neq d$ cannot be replicated.

OR

Let m, n denotes the number of rows and columns respectively. Since $m = 4 > 3 = n$, the matrix S is non-invertible, and hence the market is not complete.

- b) Assume $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)'$.

$$(\mathbf{S}^0)' = \psi' \mathbf{S}$$

$$\Rightarrow \begin{cases} 1.3\psi_1 + 0.9\psi_2 = 1 & (1) \\ 0.8\psi_1 + 1.1\psi_2 = 1 & (2) \\ 1.2\psi_3 + 2.4\psi_4 = 2 & (3) \end{cases}$$

Solving simultaneous equations above, one possible state price vector is

$$\psi = \left(\frac{20}{71}, \frac{50}{71}, \frac{1}{2}, \frac{7}{12} \right)'$$

The state price vector is not unique since we can arbitrarily define ψ_3 to get ψ_4 from Eq(3).

Remark:

- i) Suppose that there exist multiple state price vectors $\psi_i, i = 1, 2, \dots$. By definition, $(\mathbf{S}^0)' = \psi_i' \mathbf{S}$ for each $i = 1, 2, \dots$. If $\mathbf{F}$ is replicable, then $\mathbf{F} = \mathbf{S}\mathbf{w}$, and the price is

$$F^0 = (\mathbf{S}^0)' \mathbf{w} = \psi_i' \mathbf{S} \mathbf{w} = \psi_i' \mathbf{F}$$

for all $i = 1, \dots$. That is, even though there exist multiple state price vectors, they all produce the same price using the formula $\psi_i' \mathbf{F}$, if $\mathbf{F}$ is replicable.

- ii) We established that " $\mathbf{F}$ is replicable implies $\psi_i' \mathbf{F}$ are the same for all $i = 1, \dots$ ". This implies that "If $\psi_i' \mathbf{F}$ and $\psi_j' \mathbf{F}$ are different for some $i \neq j$, then $\mathbf{F}$ is not replicable."

- c) Let $\mathbf{F} = (1, 1, 1, 1)'$ be the payoff of a risk-free asset. Assume there is a risk-free asset, $\mathbf{w}$, in the market, and $\mathbf{S}\mathbf{w} = \mathbf{F}$. We have

$$1.2w_3 = 1$$

$$2.4w_3 = 1$$

which is not possible. Hence, there is no risk-free asset in the market.

- d) Since there exists a positive state price vector ($\psi \gg 0$) from results in (b), there is no arbitrage opportunity in the market.

Even though there may exist a state price vector (with negative entries) $\tilde{\psi}$ satisfying $\tilde{F}^0 = \tilde{\psi}' \mathbf{F} \leq 0$ for some $\mathbf{F} \gg 0$ (i.e. arbitrage), since $F^0 = \psi' \mathbf{F} > 0$ is different from $\tilde{F}^0$, Remark ii) in b) guarantees that this $\mathbf{F}$ is not replicable, and thus the arbitrage is not achievable.

- e) Let $\mathbf{F} = (1, 2, 4, 8)'$ and $\mathbf{w} = (w_1, w_2, w_3)'$.

$$\mathbf{S}\mathbf{w} = \mathbf{F}$$

$$\Rightarrow \begin{cases} 1.3w_1 + 0.8w_2 = 1 \\ 0.9w_1 + 1.1w_2 = 2 \\ 1.2w_3 = 4 \\ 2.4w_3 = 8 \end{cases}$$

Solving simultaneous equations above, we have

$$\mathbf{w} = \left(-\frac{50}{71}, \frac{170}{71}, \frac{10}{3} \right)'$$

- f) Let $\mathbf{F} = (1, 2, 4, 8)'$

$$\begin{aligned} F^0 &= (\mathbf{S}^0)' \mathbf{w} \\ &= 8.3568 \end{aligned}$$

OR

$$\begin{aligned} F^0 &= \psi' \mathbf{F} \\ &= 8.3568 \end{aligned}$$

6) a)

$$\begin{aligned}
 \int_0^\infty \int_{-x}^\infty \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dy dx &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^\infty \frac{1}{2\pi} e^{-\frac{1}{2}r^2} r dr d\theta \quad (\text{Let } x = r \cos \theta, y = r \sin \theta) \\
 &= \frac{3\pi}{4} \int_0^\infty \frac{1}{2\pi} r e^{-\frac{1}{2}r^2} dr \\
 &= \frac{3}{8} \left[-e^{-\frac{1}{2}r^2} \right]_0^\infty \\
 &= \frac{3}{8}
 \end{aligned}$$

b)

$$\begin{aligned}
 P(B_1 > 0, B_2 > 0) &= P(B_1 > 0, B_2 - B_1 > -B_1) \\
 &= P(Z_1 > 0, Z_2 > -Z_1) \quad (\text{where } Z_1 \sim N(0, 1), Z_2 \sim N(0, 1)) \\
 &= \int_0^\infty \int_{-z_1}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_2^2} dz_2 dz_1 \\
 &= \int_0^\infty \int_{-z_1}^\infty \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)} dz_2 dz_1 \\
 &= \frac{3}{8}
 \end{aligned}$$

7) For $s < t$,

$$\begin{aligned}
 \mathbb{E}(B_t^2 - t | \mathcal{F}_s) &= \mathbb{E}\left\{[(B_t - B_s) + B_s]^2 | \mathcal{F}_s\right\} - t \\
 &= \mathbb{E}[(B_t - B_s)^2] - 2B_s \mathbb{E}(B_t - B_s) + B_s^2 - t \quad (\text{since } B_t - B_s \perp \mathcal{F}_s) \\
 &= t - s + 0 + B_s^2 - t \quad (\text{since } B_t - B_s \sim N(0, t - s)) \\
 &= B_s^2 - s
 \end{aligned}$$

8) Recall Ito's Lemma: $df(t, B_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B_t} dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial B_t^2} (dB_t)^2$.

Let $X_t = f(t, B_t) = \sin B_t$.

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial B_t} = \cos B_t, \quad \text{and} \quad \frac{\partial^2 f}{\partial B_t^2} = -\sin B_t.$$

$$\begin{aligned} dX_t &= 0 \cdot dt - \frac{1}{2} \sin B_t dt + \cos B_t dB_t \\ &= -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2} dB_t \end{aligned}$$

Hence, $X_t = \sin B_t$ with $X_0 = \sin B_0 = 0$.

Alternative solution:

Let $f(t, X_t) = Y_t = \arcsin X_t$.

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial X_t} = \frac{1}{\sqrt{1 - X_t^2}}, \quad \text{and} \quad \frac{\partial^2 f}{\partial X_t^2} = \frac{X_t}{(1 - X_t^2)^{3/2}}.$$

$$\begin{aligned} dY_t &= \frac{1}{\sqrt{1 - X_t^2}} \left(-\frac{1}{2} X_t dt + \sqrt{1 - X_t^2} dB_t \right) + \frac{1}{2} \left(\frac{X_t}{(1 - X_t^2)^{3/2}} \right) (1 - X_t^2) dt \\ &= \left[-\frac{X_t}{2\sqrt{1 - X_t^2}} + \frac{X_t}{2\sqrt{1 - X_t^2}} \right] dt + dB_t \\ &= dB_t \end{aligned}$$

Hence,

$$\arcsin X_t - \arcsin X_0 = B_t - B_0$$

$$\arcsin X_t = k\pi + B_t$$

$$X_t = \sin(k\pi + B_t), \quad k \in \mathbb{Z}$$