

Department of Statistics, The Chinese University of Hong Kong
RMSC4003 Statistical Modelling in Financial Markets

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1 (30 points)

$$S^0 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \quad S = \begin{pmatrix} 1.3 & 1.6 & 0 & 0 \\ 0.9 & 2.2 & 0 & 0 \\ 0 & 0 & 1.2 & 2.4 \end{pmatrix}$$

a) The market is complete. **[1]**

Any payoff, F , is attainable (can be replicated using the assets available), where $F = Sw$. **[2]**

b) Asset 4 (or 3) is redundant (as the columns of S are not linearly independent). **[1]**

$$S = \begin{pmatrix} 1.3 & 1.6 & 0 \\ 0.9 & 2.2 & 0 \\ 0 & 0 & 1.2 \end{pmatrix} \quad \mathbf{[2]}$$

c)

$$(S^0)' = \psi' S \quad \mathbf{[2]} \quad \Rightarrow \quad \begin{cases} 1.3\psi_1 + 0.9\psi_2 = 1 \\ 1.6\psi_1 + 2.2\psi_2 = 2 \\ 1.2\psi_3 = 1 \end{cases} \quad \mathbf{[3]} \quad \Rightarrow \quad \psi = \begin{pmatrix} \frac{20}{71} \\ \frac{50}{71} \\ \frac{5}{6} \end{pmatrix} \quad \mathbf{[1]}$$

d)

$$\bar{\psi} = \frac{20}{71} + \frac{50}{71} + \frac{5}{6} = \frac{1}{1+r} \quad \mathbf{[2]}$$

$$\frac{775}{426} = \frac{1}{1+r} \Rightarrow r = \frac{426}{775} - 1 = -\frac{349}{775} = -45.03\% \quad \mathbf{[1]}$$

e)

$$\tilde{\psi} = \frac{\psi}{\bar{\psi}} = \left(\frac{24}{155}, \frac{12}{31}, \frac{71}{155} \right)' \quad \mathbf{[3]}$$

f) Since all entries of the unique state price vector are strictly greater than 0, that is, $\psi \gg 0$ from the results in c), there is no arbitrage opportunity in the market. **[3]**

g)

$$Sw = F \quad \mathbf{[1]} \quad \Rightarrow \quad \begin{cases} 1.3w_1 + 1.6w_2 = 7 \\ 0.9w_1 + 2.2w_2 = 2 \\ 1.2w_3 = 1 \end{cases} \quad \mathbf{[1]} \quad \Rightarrow \quad w = \begin{pmatrix} \frac{610}{71} \\ -\frac{185}{71} \\ \frac{5}{6} \end{pmatrix} \quad \mathbf{[1]}$$

h) [3]

$$\begin{aligned}
F^0 &= \boldsymbol{\psi}' \mathbf{F} = \left(\frac{20}{71} \quad \frac{50}{71} \quad \frac{5}{6} \right) \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} = \frac{2215}{426} = 5.1995 \\
F^0 &= \bar{\psi} E_{\tilde{\psi}}(\mathbf{F}) = \frac{775}{426} \left(\frac{24}{155}(8) + \frac{12}{31}(3) + \frac{71}{155} \right) = \frac{2215}{426} = 5.1995 \\
\mathbf{S}\mathbf{w} = \mathbf{F} &\implies \begin{cases} 1.3w_1 + 1.6w_2 = 8 \\ 0.9w_1 + 2.2w_2 = 3 \\ 1.2w_3 = 1 \end{cases} \implies \mathbf{w} = \begin{pmatrix} \frac{640}{71} \\ -\frac{165}{71} \\ \frac{5}{6} \end{pmatrix} \\
F^0 &= \mathbf{w}' \mathbf{S}^0 = \left(\frac{640}{71} \quad -\frac{165}{71} \quad \frac{5}{6} \right) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{2215}{426} = 5.1995
\end{aligned}$$

i) Provide an example of an arbitrage opportunity. [3]

Short one unit of $\mathbf{S}_1$, short one unit of $\mathbf{S}_2$, short one unit of $\mathbf{S}_3$ and long two units of $\mathbf{F}$. The initial investment is $1 + 2 + 1 - 2(2) = 0$.

At maturity, the payoff under scenario 1 is $-1.3 - 2(1.6) + 16 = 11.5$, the payoff under scenario 2 is $-0.9 - 2(2.2) + 6 = 0.7$ and the payoff under scenario 3 is $-1.2 + 2 = 0.8$. Hence, we have $\mathbf{w}' \mathbf{S}^0 \leq 0$ and $\mathbf{S}\mathbf{w} > \mathbf{0}$.

2 (10 points)

a) Using Taylor series expansion $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O(x^5)$,

$$\begin{aligned}
e^{r\delta} - e^{\mu\delta - \sigma\sqrt{\delta}} &= \left(1 + r\delta + \frac{r^2\delta^2}{2} + O(\delta^3) \right) - \left(1 + \mu\delta - \sigma\sqrt{\delta} + \frac{\mu^2\delta^2 - 2\mu\sigma\delta^{\frac{3}{2}} + \sigma^2\delta}{2} + \frac{3\mu\sigma^2\delta^2 - \sigma^3\delta^{\frac{3}{2}}}{6} + \frac{\sigma^4\delta^2}{24} + O(\delta^{\frac{5}{2}}) \right) \quad [1] \\
&= \sigma\sqrt{\delta} + r\delta - \mu\delta - \frac{\sigma^2}{2}\delta + \mu\sigma\delta^{\frac{3}{2}} + \frac{\sigma^3}{6}\delta^{\frac{3}{2}} + \frac{r^2}{2}\delta^2 - \frac{\mu^2}{2}\delta^2 - \frac{\sigma^4}{24}\delta^2 - \frac{\mu\sigma^2}{2}\delta^2 + O(\delta^{\frac{5}{2}}) \quad [1] \\
&= \sigma\sqrt{\delta} + \left(r - \mu - \frac{\sigma^2}{2} \right) \delta + \left(\mu\sigma + \frac{\sigma^3}{6} \right) \delta^{\frac{3}{2}} + \left(\frac{r^2}{2} - \frac{\mu^2}{2} - \frac{\sigma^4}{24} - \frac{\mu\sigma^2}{2} \right) \delta^2 + O(\delta^{\frac{5}{2}})
\end{aligned}$$

b) Using Taylor series expansion $\frac{1}{1+x} = 1 - x + O(x^2)$,

$$\begin{aligned}
e^{\mu\delta + \sigma\sqrt{\delta}} &= 1 + \mu\delta + \sigma\sqrt{\delta} + \frac{\mu^2\delta^2 + 2\mu\sigma\delta^{\frac{3}{2}} + \sigma^2\delta}{2} + \frac{3\mu\sigma^2\delta^2 + \sigma^3\delta^{\frac{3}{2}}}{6} + \frac{\sigma^4\delta^2}{24} + O(\delta^{\frac{5}{2}}) \quad [1] \\
e^{\mu\delta - \sigma\sqrt{\delta}} &= 1 + \mu\delta - \sigma\sqrt{\delta} + \frac{\mu^2\delta^2 - 2\mu\sigma\delta^{\frac{3}{2}} + \sigma^2\delta}{2} + \frac{3\mu\sigma^2\delta^2 - \sigma^3\delta^{\frac{3}{2}}}{6} + \frac{\sigma^4\delta^2}{24} + O(\delta^{\frac{5}{2}}) \\
\frac{1}{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{\mu\delta - \sigma\sqrt{\delta}}} &= \frac{1}{2\sigma\sqrt{\delta} + 2\mu\sigma\delta^{\frac{3}{2}} + \frac{1}{3}\sigma^3\delta^{\frac{3}{2}} + O(\delta^{\frac{5}{2}})} = \frac{1}{2\sigma\sqrt{\delta}} \frac{1}{1 + ((\mu + \frac{1}{6}\sigma^2)\delta + O(\delta^2))} \quad [2] \\
&= \frac{1}{2\sigma\sqrt{\delta}} \left(1 - \left(\left(\mu + \frac{1}{6}\sigma^2 \right) \delta + O(\delta^2) \right) + \left(\left(\mu + \frac{1}{6}\sigma^2 \right) \delta + O(\delta^2) \right)^2 - \dots \right) \\
&= \frac{1}{2\sigma\sqrt{\delta}} \left(1 - \left(\mu + \frac{1}{6}\sigma^2 \right) \delta + O(\delta^2) \right)
\end{aligned}$$

c)

$$\begin{aligned}
 q &= \frac{e^{r\delta} - e^{\mu\delta - \sigma\sqrt{\delta}}}{e^{\mu\delta + \sigma\sqrt{\delta}} - e^{\mu\delta - \sigma\sqrt{\delta}}} \\
 &= \frac{1}{2\sigma\sqrt{\delta}} \left[1 - \left(\mu + \frac{1}{6}\sigma^2 \right) \delta + O(\delta^2) \right] \left[\sigma\sqrt{\delta} + \left(r - \mu - \frac{\sigma^2}{2} \right) \delta + \left(\mu\sigma + \frac{\sigma^3}{6} \right) \delta^{\frac{3}{2}} \right. \\
 &\quad \left. + \left(\frac{r^2}{2} - \frac{\mu^2}{2} - \frac{\sigma^4}{24} - \frac{\mu\sigma^2}{2} \right) \delta^2 + O\left(\delta^{\frac{5}{2}}\right) \right] \\
 &= \frac{1}{2} + \frac{\left(r - \mu - \frac{\sigma^2}{2} \right) \sqrt{\delta}}{2\sigma} + \frac{\left(\mu\sigma + \frac{\sigma^3}{6} \right) \delta}{2\sigma} - \frac{\left(\mu\sigma + \frac{\sigma^3}{6} \right) \delta}{2\sigma} + \frac{\left(\frac{r^2}{2} - \frac{\mu^2}{2} - \frac{\sigma^4}{24} - \frac{\mu\sigma^2}{2} \right) \delta^{\frac{3}{2}}}{2\sigma} \\
 &\quad - \frac{\left(r - \mu - \frac{\sigma^2}{2} \right) \left(\mu + \frac{\sigma^2}{6} \right) \delta^{\frac{3}{2}}}{2\sigma} + O(\delta^2) \\
 &= \frac{1}{2} + \frac{\left(r - \mu - \frac{\sigma^2}{2} \right) \sqrt{\delta}}{2\sigma} + \frac{\left(\left(\frac{r^2}{2} - \frac{\mu^2}{2} - \frac{\sigma^4}{24} - \frac{\mu\sigma^2}{2} \right) - \left(r - \mu - \frac{\sigma^2}{2} \right) \left(\mu + \frac{\sigma^2}{6} \right) \right) \delta^{\frac{3}{2}}}{2\sigma} + O(\delta^2) \\
 &= \frac{1}{2} - \frac{\left(\mu - r + \frac{\sigma^2}{2} \right) \sqrt{\delta}}{2\sigma} + \frac{\left(\left(r^2 - \mu^2 - \frac{\sigma^4}{12} - \mu\sigma^2 \right) - (2r - 2\mu - \sigma^2) \left(\mu + \frac{\sigma^2}{6} \right) \right) \delta^{\frac{3}{2}}}{4\sigma} + O(\delta^2) \\
 &= \frac{1}{2} - \frac{\left(\mu - r + \frac{\sigma^2}{2} \right) \sqrt{\delta}}{2\sigma} + \frac{\left((\mu - r)^2 + \frac{1}{3} \left(\mu - r + \frac{1}{4}\sigma^2 \right) \sigma^2 \right) \delta^{\frac{3}{2}}}{4\sigma} + O(\delta^2)
 \end{aligned}$$

 i) $q \rightarrow \frac{1}{2}$ as $\delta \rightarrow 0$. **[0.5]**

ii)

$$\frac{2q-1}{\sqrt{\delta}} = \frac{2 \left(\frac{1}{2} + \frac{\sqrt{\delta} \left(r - \mu - \frac{1}{2}\sigma^2 \right)}{2\sigma} + O\left(\delta^{\frac{3}{2}}\right) \right) - 1}{\sqrt{\delta}} = \frac{r - \mu - \frac{1}{2}\sigma^2}{\sigma} + O(\delta) = -\frac{1}{\sigma} \left(\mu + \frac{1}{2}\sigma^2 - r \right) + O(\delta)$$

 Therefore, $\frac{2q-1}{\sqrt{\delta}} \rightarrow -\frac{1}{\sigma} \left(\mu + \frac{1}{2}\sigma^2 - r \right)$ as $\delta \rightarrow 0$. **[0.5]**

iii)

$$\begin{aligned}
 &\frac{2q-1 + \frac{1}{\sigma} \left(\mu - r + \frac{1}{2}\sigma^2 \right) \sqrt{\delta}}{\delta} \\
 &= \frac{1}{\delta} \left[-\frac{1}{\sigma} \left(\mu - r + \frac{1}{2}\sigma^2 \right) \sqrt{\delta} + \frac{1}{2\sigma} \left((\mu - r)^2 + \frac{1}{3} \left(\mu - r + \frac{1}{4}\sigma^2 \right) \sigma^2 \right) \delta^{\frac{3}{2}} \right. \\
 &\quad \left. + \frac{1}{\sigma} \left(\mu - r + \frac{1}{2}\sigma^2 \right) \sqrt{\delta} + O(\delta^2) \right] \\
 &= \frac{1}{2\sigma} \left((\mu - r)^2 + \frac{1}{3} \left(\mu - r + \frac{1}{4}\sigma^2 \right) \sigma^2 \right) \sqrt{\delta} + O(\delta) = O\left(\delta^{\frac{1}{2}}\right)
 \end{aligned}$$

 Therefore, as $\delta \rightarrow 0$, the term goes to 0. **[1.5]**

iv)

$$\begin{aligned}
 & \frac{2q-1+\frac{1}{\sigma}\left(\mu-r+\frac{1}{2}\sigma^2\right)\sqrt{\delta}}{\delta^{\frac{3}{2}}} \\
 &= \frac{1}{\delta^{\frac{3}{2}}} \left[-\frac{1}{\sigma} \left(\mu-r+\frac{1}{2}\sigma^2 \right) \sqrt{\delta} + \frac{1}{2\sigma} \left((\mu-r)^2 + \frac{1}{3} \left(\mu-r+\frac{1}{4}\sigma^2 \right) \sigma^2 \right) \delta^{\frac{3}{2}} \right. \\
 & \quad \left. + \frac{1}{\sigma} \left(\mu-r+\frac{1}{2}\sigma^2 \right) \sqrt{\delta} + O(\delta^2) \right] \\
 &= \frac{1}{2\sigma} \left((\mu-r)^2 + \frac{1}{3} \left(\mu-r+\frac{1}{4}\sigma^2 \right) \sigma^2 \right) + O(\delta^{\frac{1}{2}}) \\
 & \frac{2q-1+\frac{1}{\sigma}\left(\mu-r+\frac{1}{2}\sigma^2\right)\sqrt{\delta}}{\delta^{\frac{3}{2}}} \rightarrow \frac{1}{2\sigma} \left((\mu-r)^2 + \frac{1}{3} \left(\mu-r+\frac{1}{4}\sigma^2 \right) \sigma^2 \right) \text{ as } \delta \rightarrow 0. \quad \mathbf{[1.5]}
 \end{aligned}$$

3 (5 points)

By Ito's lemma,

$$d(e^{ct}W_t) = e^{ct}dW_t + ce^{ct}W_tdt \quad \mathbf{[4]}$$

$$e^{cT}W_T = \int_0^T e^{ct}dW_t + \int_0^T ce^{ct}W_tdt$$

$$\underbrace{\int_0^T e^{ct}dW_t}_{\text{Stochastic integral}} = e^{cT}W_T - \underbrace{\int_0^T ce^{ct}W_tdt}_{\text{Riemann integral}} \quad \mathbf{[1]}$$

4 (15 points)

$$F^0 = e^{-rT} \mathbf{E}_{\tilde{\Psi}} (2 \cdot \mathbf{1}_{(S_T \geq 110)} + \mathbf{1}_{(S_T \geq 120)}) = e^{-10r} (2\mathbf{E}_{\tilde{\Psi}} (\mathbf{1}_{(S_{10} \geq 110)}) + \mathbf{E}_{\tilde{\Psi}} (\mathbf{1}_{(S_{10} \geq 120)})) \quad \mathbf{[5]}$$

$$= e^{-10r} (2\mathbb{P}_{\tilde{\Psi}}(S_{10} \geq 110) + \mathbb{P}_{\tilde{\Psi}}(S_{10} \geq 120)) \quad \mathbf{[3]}$$

$$= e^{-10r} \left(2\mathbb{P} \left[S_0 \exp \left(10 \left(r - \frac{1}{2}\sigma^2 \right) + \sigma N(0, T) \right) \geq 110 \right] + \mathbb{P} \left[S_0 \exp \left(10 \left(r - \frac{1}{2}\sigma^2 \right) + \sigma N(0, T) \right) \geq 120 \right] \right)$$

$$= e^{-10r} \left(2\mathbb{P} \left[\frac{N(0, 10)}{\sqrt{10}} \geq -\frac{\ln \left(\frac{100}{110} \right) + 10 \left(0.02 - \frac{1}{2}(0.1)^2 \right)}{0.1\sqrt{10}} \right] + \mathbb{P} \left[\frac{N(0, 10)}{\sqrt{10}} \geq -\frac{\ln \left(\frac{100}{120} \right) + 10 \left(0.02 - \frac{1}{2}(0.1)^2 \right)}{0.1\sqrt{10}} \right] \right)$$

$$d_2^1 = \frac{\ln \left(\frac{100}{110} \right) + 10 \left(0.02 - \frac{1}{2}(0.1)^2 \right)}{0.1\sqrt{10}} = 0.172944 \quad d_2^2 = \frac{\ln \left(\frac{100}{120} \right) + 10 \left(0.02 - \frac{1}{2}(0.1)^2 \right)}{0.1\sqrt{10}} = -0.102210 \quad \mathbf{[3]}$$

$$= 2e^{-10(0.02)} N(d_2^1) + e^{-10(0.02)} N(d_2^2) \quad \mathbf{[4]}$$

$$= 1.3072$$

5 (15 points)

a)

$$\text{Cov} \left(B_{\frac{1}{2}} - \frac{1}{2}B_1, B_1 \right) = \text{Cov} \left(B_{\frac{1}{2}}, B_1 \right) - \frac{1}{2}\text{Cov}(B_1, B_1) = \frac{1}{2} - \frac{1}{2} = 0 \quad \mathbf{[3]}$$

b)

$$\begin{aligned} \text{Cov}(B_t - xB_1, B_1) &= \text{Cov}(B_t, B_1) - x\text{Cov}(B_1, B_1) = \min(t, 1) - x = 0 \quad [2] \\ x &= \min(t, 1) \quad [1] \end{aligned}$$

c)

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \quad [1] \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2 - 2t\sigma^2 x}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2 - 2t\sigma^2 x + \sigma^4 t^2 - \sigma^4 t^2}{2\sigma^2}\right) dx \quad [1] \\ &= \exp\left(-\frac{\sigma^4 t^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \sigma^2 t)^2}{2\sigma^2}\right) dx \quad [1] \\ &= \exp\left(\frac{1}{2}\sigma^2 t^2\right) \end{aligned}$$

d)

$$\begin{aligned} \mathbb{E}\left(e^{aB_1} \middle| B_{\frac{1}{2}}\right) &= \mathbb{E}\left(e^{a\left(B_1 - B_{\frac{1}{2}}\right) + aB_{\frac{1}{2}}} \middle| B_{\frac{1}{2}}\right) \quad [1] \\ &= e^{aB_{\frac{1}{2}}} \mathbb{E}\left(e^{a\left(B_1 - B_{\frac{1}{2}}\right)} \middle| B_{\frac{1}{2}}\right) \quad [1] \\ &= e^{aB_{\frac{1}{2}}} e^{\frac{1}{2}a^2} = \exp\left(\frac{a^2}{4} + aB_{\frac{1}{2}}\right) \quad [1] \end{aligned}$$

e)

$$\begin{aligned} \mathbb{E}\left(e^{aB_{\frac{1}{2}}} \middle| B_1\right) &= \mathbb{E}\left(e^{a\left(B_{\frac{1}{2}} - \frac{1}{2}B_1\right) + \frac{a}{2}B_1} \middle| B_1\right) \quad [1] \\ &= e^{\frac{a}{2}B_1} \mathbb{E}\left(e^{a\left(B_{\frac{1}{2}} - \frac{1}{2}B_1\right)} \middle| B_1\right) \\ &= e^{\frac{a}{2}B_1} \mathbb{E}\left(e^{a\left(B_{\frac{1}{2}} - \frac{1}{2}B_1\right)}\right) \quad \text{since } B_{\frac{1}{2}} - \frac{1}{2}B_1 \text{ and } B_1 \text{ are independent} \quad [1] \\ &= e^{\frac{a}{2}B_1} \mathbb{E}\left(\mathbb{E}\left(e^{a\left(B_{\frac{1}{2}} - \frac{1}{2}B_1\right)} \middle| B_{\frac{1}{2}}\right)\right) = e^{\frac{a}{2}B_1} \mathbb{E}\left(e^{\frac{a}{2}B_{\frac{1}{2}}} \mathbb{E}\left(e^{-\frac{a}{2}\left(B_1 - B_{\frac{1}{2}}\right)} \middle| B_{\frac{1}{2}}\right)\right) \\ &= e^{\frac{a}{2}B_1} \mathbb{E}\left(e^{\frac{a}{2}B_{\frac{1}{2}}} e^{\frac{a^2}{16}}\right) = e^{\frac{a}{2}B_1} e^{\frac{a^2}{16}} \mathbb{E}\left(e^{\frac{a}{2}B_{\frac{1}{2}}}\right) = e^{\frac{a}{2}B_1} e^{\frac{a^2}{16}} e^{\frac{a^2}{16}} \\ &= \exp\left(\frac{a^2}{8} + \frac{a}{2}B_1\right) \quad [1] \end{aligned}$$

6 (10 points)

For $s < t$:

$$\begin{aligned}
 \mathbb{E}(X_t | \mathcal{F}_s) &= \mathbb{E} \left(e^{aW_s - \frac{a^2 s}{2} + a(W_t - W_s) - \frac{a^2(t-s)}{2}} \middle| \mathcal{F}_s \right) \quad [3] \\
 &= \mathbb{E} \left(X_s \cdot e^{a(W_t - W_s) - \frac{a^2(t-s)}{2}} \middle| \mathcal{F}_s \right) \\
 &= X_s \cdot \underbrace{\mathbb{E} \left(e^{a(W_t - W_s) - \frac{a^2(t-s)}{2}} \middle| \mathcal{F}_s \right)}_{(*) = 1} \quad [3] \\
 &= X_s
 \end{aligned}$$

Two ways to prove $(*) = 1$. **[4]**

Approach 1: Recall MGF: $M(t) = \mathbb{E}(e^{tX}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$. Hence, $\mathbb{E}(e^{a(W_t - W_s)} | \mathcal{F}_s) = e^{\frac{1}{2}(t-s)a^2}$.

$$\mathbb{E} \left(e^{a(W_t - W_s) - \frac{a^2(t-s)}{2}} \middle| \mathcal{F}_s \right) = e^{\frac{a^2}{2}(t-s)} \cdot e^{-\frac{a^2}{2}(t-s)} = 1$$

Approach 2: If $Z \sim (0, h)$, then $\mathbb{E} \left(e^{z - \frac{h}{2}} \right) = 1$.

$$\int e^{z - \frac{h}{2}} \frac{1}{\sqrt{2\pi h}} e^{-\frac{z^2}{2h}} dz = \int \frac{1}{\sqrt{2\pi h}} e^{-\frac{(z-h)^2}{2h}} dz = 1$$

7 (15 points)

a) By Ito's lemma,

$$df(t, X_t) = \left(\frac{\partial f(t, x)}{\partial t} + a \frac{\partial f(t, x)}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 f(t, x)}{\partial x^2} \right) dt + b \frac{\partial f(t, x)}{\partial x} dB_t \quad [1]$$

$$= \left(\frac{\partial f(t, X_t)}{\partial t} + \frac{1}{2} \frac{\partial f(t, X_t)}{\partial X_t} + \frac{1}{2} X_t \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \right) dt + \sqrt{X_t} \frac{\partial f(t, X_t)}{\partial X_t} dB_t \quad [2]$$

b)

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial f}{\partial x} + \frac{x}{2} \frac{\partial^2 f}{\partial x^2} = 0 \quad [3]$$

c) i) X_t is not a martingale since $dX_t = \frac{1}{2}dt + \sqrt{X_t}dB_t$. **[2]**

$$X_t = X_s + \frac{1}{2}(t-s) + \int_s^t \sqrt{X_\tau} dB_\tau$$

$$\mathbb{E}(X_t | \mathcal{F}_s) = X_s + \frac{1}{2}(t-s) \neq X_s$$

ii) By Ito's lemma,

$$\begin{aligned}
 d(t - 2X_t) &= \left(\frac{\partial f(t, X_t)}{\partial t} + \frac{1}{2} \frac{\partial f(t, X_t)}{\partial X_t} + \frac{1}{2} X_t \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \right) dt + \sqrt{X_t} \frac{\partial f(t, X_t)}{\partial X_t} dB_t \\
 &= \left(1 + \frac{1}{2}(-2) + 0 \right) dt + \sqrt{X_t}(-2)dB_t = -2\sqrt{X_t}dB_t \quad [2]
 \end{aligned}$$

$$Z_t = t - 2X_t \implies Z_t = Z_s - \int_s^t 2\sqrt{X_\tau} dB_\tau \implies \mathbb{E}(Z_t | \mathcal{F}_s) = Z_s$$

Hence, $t - 2X_t$ is a martingale.

iii) By Ito's lemma,

$$\begin{aligned}
 d \log X_t &= \left(\frac{\partial f(t, X_t)}{\partial t} + \frac{1}{2} \frac{\partial f(t, X_t)}{\partial X_t} + \frac{1}{2} X_t \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \right) dt + \sqrt{X_t} \frac{\partial f(t, X_t)}{\partial X_t} dB_t \\
 &= \left(0 + \frac{1}{2} \frac{1}{X_t} - \frac{1}{2} X_t \frac{1}{X_t^2} \right) dt + \sqrt{X_t} \frac{1}{X_t} dB_t = \left(\frac{1}{2} \frac{1}{X_t} - \frac{1}{2} \frac{1}{X_t} \right) dt + \frac{1}{\sqrt{X_t}} dB_t \\
 &= \frac{1}{\sqrt{X_t}} dB_t \quad [2] \\
 \log X_t &= \log X_s + \int_s^t \frac{1}{\sqrt{X_\tau}} dB_\tau \implies E(\log X_t | \mathcal{F}_s) = \log X_s
 \end{aligned}$$

Hence, $\log X_t$ is a martingale.

d) $\tau = \min\{t \geq 0 : X_t = e \text{ or } X_t = e^{-1}\} = \min\{t \geq 0 : \log X_t = 1 \text{ or } \log X_t = -1\}$.

Because $\log X_t$ is a martingale and $\log X_0 = 0$, by OST, [1]

$$\begin{aligned}
 0 &= E(\log X_\tau) = 1 \cdot \mathbb{P}(\log X_\tau = 1) - 1 \cdot \mathbb{P}(\log X_\tau = -1) \quad [2] \\
 \implies &\begin{cases} \mathbb{P}(X_\tau = e) - \mathbb{P}(X_\tau = e^{-1}) = 0 \\ \mathbb{P}(X_\tau = e) + \mathbb{P}(X_\tau = e^{-1}) = 1 \end{cases} \\
 \mathbb{P}(X_\tau = e) &= \mathbb{P}(X_\tau = e^{-1}) = \frac{1}{2}
 \end{aligned}$$

Because $t - 2X_t$ is a martingale and $0 - 2X_0 = -2$, by OST,

$$\begin{aligned}
 -2 &= E(\tau - 2X_\tau) \\
 E(\tau) &= 2E(X_\tau) - 2 = 2(e\mathbb{P}(X_\tau = e) + e^{-1}\mathbb{P}(X_\tau = e^{-1})) - 2 \\
 E(\tau) &= 2\left(e\frac{1}{2} + e^{-1}\frac{1}{2}\right) - 2 = e + e^{-1} - 2
 \end{aligned}$$