

香 港 中 文 大 學
The Chinese University of Hong Kong

Course Examinations, 1st Term, 2024-25 年度下學期科目考試

Course Code & Title 科目編號及名稱: RMSC4003: Statistical Modeling in Financial Markets

Time allowed 時間: 2 hours 小時 0 minutes 分鐘

Student I.D. No. 學生編號: _____ Seat No. 座號: _____

Answer all the questions

1) (36 marks) Consider a financial market consisting of three assets with the following structure:

- Initially, the values of the assets are $\mathbf{S}^0 = (S_1^0, S_2^0, S_3^0)' = (1, 2, 1)'$.
- At the maturity time T , there are four scenarios. The possible values are collected as

$$\mathbf{S} = \begin{pmatrix} 1.3 & 1.6 & 0 \\ 0.9 & 2.2 & 0 \\ 0 & 0 & 1.2 \\ 0 & 0 & 0.8 \end{pmatrix}, \quad (1)$$

where the (i, j) entry of $\mathbf{S}$ is the value of Stock j in the i -th scenario.

- a) (3 marks) Is the market complete? Explain.
 - b) (9 marks) Define a state price vector as a vector $\boldsymbol{\psi}$ that satisfies $(\mathbf{S}^0)' = \boldsymbol{\psi}'\mathbf{S}$.
 - i. (3 marks) Find a state price vector $\boldsymbol{\psi}$ for the market that satisfies $\boldsymbol{\psi} \gg 0$.
 - ii. (3 marks) Find a state price vector $\boldsymbol{\psi}$ for the market that contains a negative entry.
 - iii. (3 marks) Show that, for any derivative with payoff $\mathbf{F} = (F^1, \dots, F^4)$ that is replicable by the three assets, using any state price vector $\boldsymbol{\psi}$ will give the same price for $\mathbf{F}$.
 - c) (3 marks) Is the risk-free asset available in this market? Explain.
 - d) (3 marks) Show that, if there is no arbitrage opportunity in the market, then any (ω_1, ω_2) satisfying $\omega_1 \begin{pmatrix} 1.3 \\ 0.9 \end{pmatrix} + \omega_2 \begin{pmatrix} 1.6 \\ 2.2 \end{pmatrix} \geq 0$ should satisfy $\omega_1 + 2\omega_2 \geq 0$.
 - e) (3 marks) Suppose at the moment we ignore the fourth scenario in the market. Show that there is no arbitrage opportunity in the market.
 - f) (3 marks) Find the risk-neutral probabilities for the market in e).
 - g) (3 marks) Using d) and e), or otherwise, determine whether there is an arbitrage opportunity in the market in (1).
 - h) (3 marks) Find a replicating portfolio of payoffs taking values $(5, 1, 2.4, 1.6)$ in the four scenarios.
 - i) (3 marks) Price a derivative which has a payoffs $(5, 1, 2.4, 1.6)$ in the three scenarios.
 - j) (3 marks) If the derivative in i) is sold at \$ 2, explain what you will do to enjoy an arbitrage.
- 2) (4 marks) Let $q = \frac{e^{r\delta} - e^{2\mu\delta - \sigma\sqrt{\delta}}}{e^{2\mu\delta + \sigma\sqrt{\delta}} - e^{2\mu\delta - \sigma\sqrt{\delta}}}$.
- a) (2 marks) Find the limit of q as $\delta \rightarrow 0$.
 - b) (2 marks) Find the limit of $\frac{2q-1}{\sqrt{\delta}}$ as $\delta \rightarrow 0$.
- 3) (5 marks) Suppose that $dX_t = (bX_t + c)dt + 4X_t dW_t$, find an expression for dY_t in terms of dt , Y_t and dW_t , where $Y_t = \sqrt{X_t}$.
- 4) (15 marks) Let the stock price S_t follows a Black-Scholes model with risk-free rate $r = 0.04$ and volatility $\sigma = 0.12$. At time 0, $S_0 = 100$. Consider a put option which pays $(220 - 2S_{10})^+$ at time 10. Price the option.

5) (40 marks) Let $\{W_t\}$ be a Wiener process, and $\tilde{W}_k = W_{\frac{kT}{n}}$ for $k = 0, 1, 2, \dots, n$. Let

$$X_n = \sum_{k=0}^{n-1} \tilde{W}_k (\tilde{W}_{k+1} - \tilde{W}_k)^2, \quad Y_n = \frac{T}{n} \sum_{k=0}^{n-1} \tilde{W}_k, \quad P_n = \sum_{k=0}^{n-1} \tilde{W}_k^2 (\tilde{W}_{k+1} - \tilde{W}_k), \quad Q_n = \sum_{k=0}^{n-1} \tilde{W}_{k+1}^2 (\tilde{W}_{k+1} - \tilde{W}_k).$$

- (5 marks) Show that $W_T^4 = 4 \int_0^T W_t^3 dW_t + 6 \int_0^T W_t^2 dt$.
- (5 marks) Show that $E(W_t^4) = 3t^2$.
- (5 marks) Find $E(X_n)$ and $E(Y_n)$.
- (5 marks) Show that $E(X_n - Y_n)^2 \rightarrow 0$ as $n \rightarrow \infty$.
- (5 marks) Show that $\tilde{W}_k^2 (\tilde{W}_{k+1} - \tilde{W}_k) = \frac{1}{3} (\tilde{W}_{k+1}^3 - \tilde{W}_k^3) - \tilde{W}_k (\tilde{W}_{k+1} - \tilde{W}_k)^2 - \frac{1}{3} (\tilde{W}_{k+1} - \tilde{W}_k)^3$.
- (5 marks) Show that $\sum_{k=0}^{n-1} (\tilde{W}_{k+1} - \tilde{W}_k)^3 \rightarrow 0$ as $n \rightarrow \infty$.
- (5 marks) Using the above results, show from first principle that $P_n \rightarrow \frac{1}{3} W_T^3 - \int_0^T W_t^2 dt$ as $n \rightarrow \infty$.
- (5 marks) Find the limit $\lim_{n \rightarrow \infty} Q_n$.

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