

香 港 中 文 大 學
The Chinese University of Hong Kong

Course Examinations, 1st Term, 2025-26 年度下學期科目考試

Course Code & Title 科目編號及名稱: RMSC4003: Statistical Modeling in Financial Markets

Time allowed 時間: 2 hours 小時 0 minutes 分鐘

Student I.D. No. 學生編號: _____ Seat No. 座號: _____

Answer all the questions

1) (20 marks) Consider a financial market consisting of four assets with the following structure:

- Initially, the values of the assets are $\mathbf{S}^0 = (S_1^0, S_2^0, S_3^0, S_4^0)' = (1, 1, 3, 4)'$.
- At the maturity time T , there are four scenarios. The possible values are collected as

$$\mathbf{S} = \begin{pmatrix} 10.3 & 10.1 & 0 & 0 \\ 10.2 & 10.2 & 0 & 0 \\ 0 & 0 & 5.2 & 4.9 \\ 0 & 0 & 2.4 & 6.4 \end{pmatrix},$$

where the (i, j) entry of $\mathbf{S}$ is the value of Stock j in the i -th scenario.

- (3 marks) Is the market complete? Explain.
- (4 marks) Recall that a state price vector is a vector $\boldsymbol{\psi}$ which satisfies $(\mathbf{S}^0)' = \boldsymbol{\psi}'\mathbf{S}$. Find a state price vector for the market. Is the state price vector unique?
- (4 marks) Is there any risk-free asset in the market? Explain. If yes, find the risk free rate in the market.
- (3 marks) Is there any arbitrage opportunity in the market? If no, prove it. Otherwise, give an example of arbitrage portfolio.
- (3 marks) Find a replicating portfolio of payoffs taking values $(1, 2, 4, 8)$ in the four scenarios.
- (3 marks) Price a derivative which has a payoffs $(1, 2, 4, 8)$ in the four scenarios.

2) (10 marks) Let $q(\delta) = \frac{e^{\mu\delta^2 - \sigma\delta} - e^{r\delta^2}}{e^{\mu\delta^2 - \sigma\delta} - e^{\mu\delta^2 + \sigma\delta}}$.

- (3 marks) Find the limit $L = \lim_{\delta \rightarrow 0} q(\delta)$.
- (4 marks) Design a function $f(\cdot)$ such that the limit $L_1 = \lim_{\delta \rightarrow 0} \frac{q(\delta) - L}{f(\delta)}$ exists and is non-zero. Find L_1 .
- (3 marks) Design a function $g(\cdot)$ such that the limit

$$L_2 = \lim_{\delta \rightarrow 0} \frac{\frac{q(\delta) - L}{f(\delta)} - L_1}{g(\delta)}$$

exists and is non-zero.

- (5 marks) Find an alternative expression of $\int_0^T e^{W_t + t^2} dW_t$ which involves only a Riemann integral but not a stochastic integral.
- (15 marks) Let the stock price S_t follow a Black-Scholes model with risk-free rate $r = 0.01$ and volatility $\sigma = 0.1$. At time 0, $S_0 = 100$. Consider an option which is worth $6 - (S_{10} - 140)^+$ at time 10, where $X^+ = \max(0, X)$. Price the option.

5) (10 marks) Consider the Ornstein Uhlenbeck process

$$X_t = \sigma e^{ct} \int_0^t e^{-cs} dB_s,$$

where B_t is a standard Brownian motion and $c > 0$.

- (2 marks) Find $E(X_t)$
 - (4 marks) Find $\text{Var}(X_t)$, where $t > 0$.
 - (4 marks) Find $\text{Cov}(X_t, X_{t+s})$, where $s, t > 0$.
- 6) (10 marks) Let $\{B_t\}$ be a standard Brownian motion. Determine if $X_t = B_t^3 - 3tB_t$ is a martingale.
- 7) (10 marks) Let $\{B_t\}$ be a standard Brownian motion. Solve the stochastic differential equation

$$dX_t = 15X_t^{\frac{2}{3}} dt + 6X_t^{\frac{5}{6}} dB_t,$$

with $X_0 = 0$. Hint: try a B_t^k for some positive integer k .

8) (20 marks) Consider a stochastic process

$$X_t = Z_0 \frac{t}{\sqrt{2\pi}} + \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} Z_n \frac{\sin(nt/2)}{n},$$

where $Z_n \stackrel{i.i.d.}{\sim} N(0, 1)$. Note that $2 \sin(A) \sin(B) = \cos(A - B) - \cos(A + B)$.

- (2 marks) Find $E(X_t)$.
- (4 marks) Using the fact that

$$\sum_{n=1}^{\infty} \frac{\sin^2(nx)}{n^2} = \frac{\pi x}{2} - \frac{x^2}{2}, \quad (1)$$

find $\text{Var}(X_t)$.

- (4 marks) From (1) and the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$.
- (4 marks) Find $\text{Cov}(X_t, X_{t+s})$, where $s, t > 0$.
- (4 marks) Is X_t continuous w.r.t. t ? Is X_t differentiable w.r.t. t ? Explain.
- (2 marks) Which commonly used stochastic process has the same probabilistic structure as $\{X_t\}$?

— END OF THE PAPER —