

RMSC 4003 Statistical Modelling in Financial Markets (Term 1, 2022-23)

MIDTERM SOLUTION

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Part I: Applications:

1. (a) Simple returns: $RA = \exp(LA)-1$, $RB = \exp(LB)-1$, $RC = \exp(LC)-1$
(b) `mean(RB)`
(c) `cov(RA,RB)`
(d) `solve(cov(cbind(RA,RB,RC)))`
2. (a) Compute parameters:

$$\begin{aligned}c_{11} &= \mathbf{1}^T \hat{\Sigma}^{-1} \mathbf{1} = 0.9702, \\c_{1\mu} &= \mathbf{1}^T \hat{\Sigma}^{-1} \hat{\mu} = 0.03630, \\c_{\mu\mu} &= \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu} = 0.001608, \\ \Delta &= c_{11}c_{\mu\mu} - c_{1\mu}^2 = 0.0002422.\end{aligned}$$

Set the target expected return as $\mu^* = 0.045$, the variance-minimized portfolio is

$$\begin{aligned}\mathbf{w}_* &= \Delta^{-1}(c_{11}\hat{\Sigma}^{-1}\hat{\mu} - c_{1\mu}\hat{\Sigma}^{-1}\mathbf{1})\mu_* + \Delta^{-1}(c_{\mu\mu}\hat{\Sigma}^{-1}\mathbf{1} - c_{1\mu}\hat{\Sigma}^{-1}\hat{\mu}) \\ &= (-0.128, 0.674, 0.454)^T\end{aligned}$$

The volatility is

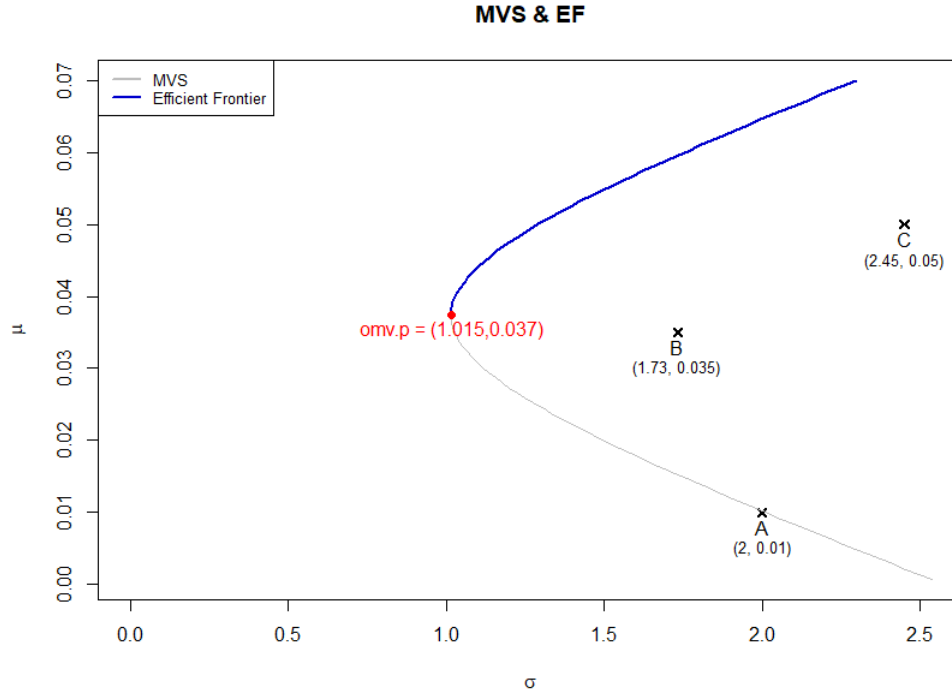
$$\sigma_* = \sqrt{\mathbf{w}_*^T \hat{\Sigma} \mathbf{w}_*} = 1.1223.$$

- (b) The expected return and volatility of the overall MVP can be found by

$$\mu_{minvar} = \frac{c_{1\mu}}{c_{11}} = 0.03741, \quad \text{and} \quad \sigma_{minvar} = \sqrt{\frac{1}{c_{11}}} = \sqrt{1.030715} = 1.0152.$$

The efficient frontier is the upper part of the minimum-variance set, that is:

$$\begin{aligned}\sigma_*^2(\mu_*) &= \Delta^{-1} \left[c_{11} \left(\mu_* - \frac{c_{1\mu}}{c_{11}} \right)^2 + c_{\mu\mu} - \frac{c_{1\mu}^2}{c_{11}} \right] \\ &= 4004.63 \times (\mu_* - 0.03741)^2 + 1.0307, \quad \mu_* > 0.03741\end{aligned}$$



(c)

$$\mathbf{w}_{\text{omvp}} = \frac{\Sigma^{-1}\mathbf{1}}{c_{11}} = (0.1161, 0.5298, 0.3542)^T$$

(d) Let $\mathbf{M}$ and $\mathbf{C}$ denotes the mean vector and covariance matrix respectively. Under no short-selling constraint, we need to apply quadratic programming to solve for the optimal portfolio with expected return $\mu_* = 0.045$.

```
library(quadprog)
V1 = matrix(1,3,1)
V0 = matrix(0,3,1)
Amat = cbind(M,V1,diag(3))
b0 = matrix(c(0.045,1,V0),3+2,1)
p = solve.QP(C, V0, Amat, b0, 2)
W0 = p$solution
```

(e) Range of target return is constrained to $[0.01, 0.05]$. This is because not allowing short sales in the assets limits the set of feasible portfolios. It is not possible to invest in a portfolio with a lower expected return than asset A and a higher expected return than asset C .

(f) The tangency portfolio is given by

$$\mathbf{w}_M = \frac{\hat{\Sigma}^{-1}(\hat{\boldsymbol{\mu}} - r\mathbf{1})}{c_{1\mu} - rc_{11}} = (-0.3589, 0.8107, 0.5481)^T,$$

$$\text{Expected return: } \mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}} = 0.05219,$$

$$\text{Volatility: } \sigma_M = \frac{|\gamma|}{c_{1\mu} - rc_{11}} = 1.3805,$$

where $\gamma^2 = (\boldsymbol{\mu} - r\mathbf{1})^T \Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1}) = 0.0005438$.

3. (a) $\text{Cov}(R_p, R_q) = \text{Cov}(\mathbf{w}_p^T \boldsymbol{\mu}, \mathbf{w}_q^T \boldsymbol{\mu}) = \mathbf{w}_p^T \hat{\Sigma} \mathbf{w}_q = 0.867$

(b) $\beta_p = \frac{\text{Cov}(R_p, R_M)}{\sigma_M^2} = \frac{\mathbf{w}_p^T \hat{\Sigma} \mathbf{w}_M}{\sigma_M^2} = 0.4199;$

(c) The security market line:

$$\mu = r + (\mu_M - r)\beta = 0.02 + 0.0322\beta.$$

The beta for portfolio q and the three other assets are:

$$\beta_q = \frac{\mathbf{w}_q^T \hat{\Sigma} \mathbf{w}_M}{\sigma_M^2} = 0.1563,$$

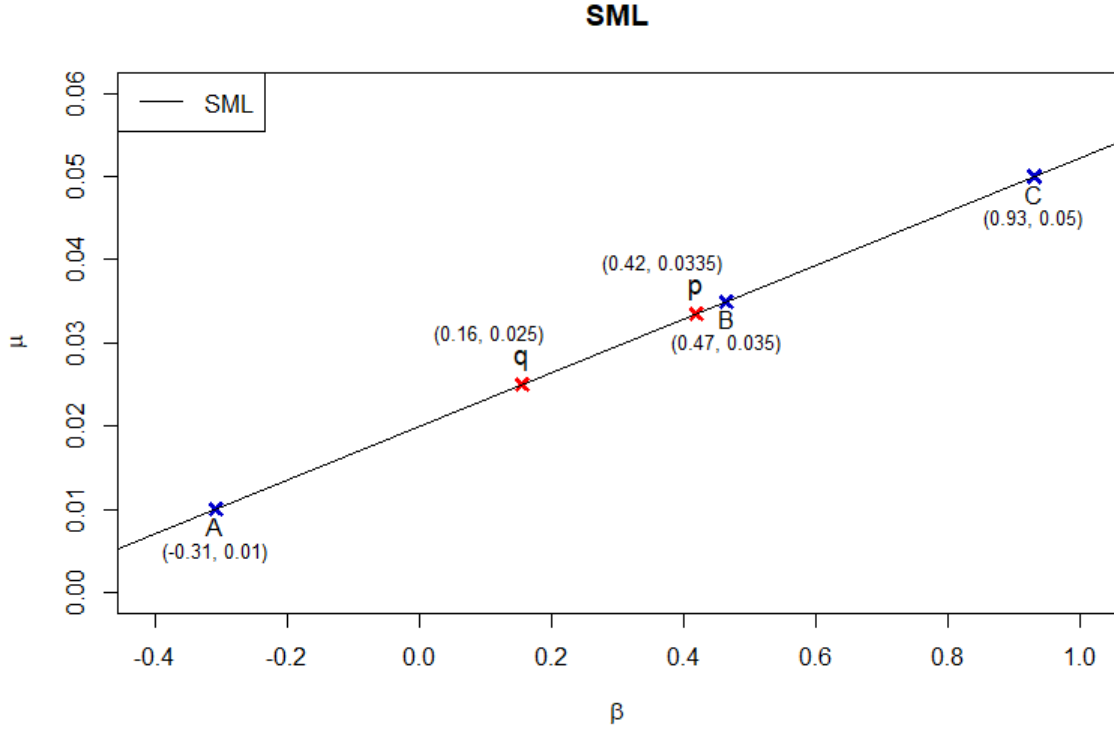
$$\beta_A = \frac{\mathbf{w}_1^T \hat{\Sigma} \mathbf{w}_M}{\sigma_M^2} = -0.3077,$$

$$\beta_B = \frac{\mathbf{w}_2^T \hat{\Sigma} \mathbf{w}_M}{\sigma_M^2} = 0.4656,$$

$$\beta_C = \frac{\mathbf{w}_3^T \hat{\Sigma} \mathbf{w}_M}{\sigma_M^2} = 0.9313,$$

where $\mathbf{w}_1 = (1, 0, 0)^T$, $\mathbf{w}_2 = (0, 1, 0)^T$, $\mathbf{w}_3 = (0, 0, 1)^T$.

$$\mu_p = \mathbf{w}_p^T \boldsymbol{\mu} = 0.0335 \quad \text{and} \quad \mu_q = \mathbf{w}_q^T \boldsymbol{\mu} = 0.025$$



(d) We have $\sigma_q = \sqrt{\mathbf{w}_q^T \hat{\Sigma} \mathbf{w}_q} = 1.5232$

$$\text{Sharpe ratio: } \frac{\mu_q - r}{\sigma_q} = \frac{0.025 - 0.02}{1.5232} = 0.003283$$

$$\text{Treynor ratio: } \frac{\mu_q - r}{\beta_p} = \frac{0.025 - 0.02}{0.1563} = 0.0320$$

(e) The factor risk and non-systematic risk for portfolio p is:

- Factor risk: $\beta_p^2 \sigma_M^2 = 0.3360$;
- Non-systematic risk: $\mathbf{w}_p^T \Sigma \mathbf{w}_p - \beta_p^2 \sigma_M^2 = 1.2630 - 0.3360 = 0.9270$.

Part II: Theory:

4. (a) $\text{Var}(\mathbf{R}) = \text{Var}(\mathbf{w}^T \boldsymbol{\mu}) = \mathbf{w}^T \text{Var}(\boldsymbol{\mu}) \mathbf{w} = \mathbf{w}^T \Sigma \mathbf{w}$

(b) From lecture notes,

$$\begin{aligned} \sigma^2(\mu_*) &= \Delta^{-1}(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu}) \\ &= \Delta^{-1}\left[c_{11}\left(\mu_* - \frac{c_{1\mu}}{c_{11}}\right)^2 + c_{\mu\mu} - \frac{c_{1\mu}^2}{c_{11}}\right] \end{aligned}$$

where

- $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$
- $c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \mathbf{1}$
- $c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$
- $\Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$

(c) Here we define the Maximum Expectation Portfolio (MEP).

$$\text{MEP} : \max_{\mathbf{w} \in \mathbb{R}^n} \mathbb{E}\left(\sum_{i=1}^n w_i R_i\right) \quad \text{subject to} \quad \text{Var}\left(\sum_{i=1}^n w_i R_i\right) = \tilde{\sigma}^2 \quad \text{and} \quad \sum_{i=1}^n w_i = 1$$

Define Lagrangian as:

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}^T \boldsymbol{\mu} - \frac{1}{2\lambda_1}(\mathbf{w}^T \Sigma \mathbf{w} - \tilde{\sigma}^2) + \lambda_2(\mathbf{w}^T \mathbf{1} - 1)$$

By comparison, we have

$$XXX = \mathbf{w}^T \Sigma \mathbf{w} - \tilde{\sigma}^2 \quad YYY = \mathbf{w}^T \mathbf{1} - 1$$

(d) Taking derivatives of Lagrangian, we get

$$\frac{\partial L}{\partial \mathbf{w}} = \boldsymbol{\mu} - \frac{1}{\lambda_1}(\Sigma \mathbf{w}) + \lambda_2 \mathbf{1} = 0 \quad (1)$$

$$\frac{\partial L}{\partial \lambda_1} = \frac{1}{2\lambda_1^2}(\mathbf{w}^T \Sigma \mathbf{w} - \tilde{\sigma}^2) = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda_2} = \mathbf{w}^T \mathbf{1} - 1 = 0 \quad (3)$$

From Eq1:

$$\begin{aligned} \frac{1}{\lambda_1} \Sigma \mathbf{w}_* &= \boldsymbol{\mu} + \lambda_2 \mathbf{1} \\ \mathbf{w}_* &= \lambda_1 \Sigma^{-1}(\boldsymbol{\mu} + \lambda_2 \mathbf{1}) \end{aligned}$$

(e) From Eq1, consider:

$$\boldsymbol{\mu}^T \Sigma^{-1} \times (1) : \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu} - \frac{1}{\lambda_1} \boldsymbol{\mu}^T \mathbf{w}_* + \lambda_2 \boldsymbol{\mu}^T \Sigma^{-1} \mathbf{1} = c_{\mu\mu} - \frac{1}{\lambda_1} \tilde{\mu} + \lambda_2 c_{\mu 1} = 0 \quad (4)$$

$$\mathbf{1}\Sigma^{-1} \times (1) : \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu} - \frac{1}{\lambda_1} \mathbf{1}^T \mathbf{w}_* + \lambda_2 \mathbf{1}^T \Sigma^{-1} \mathbf{1} \stackrel{(3)}{=} c_{1\mu} - \frac{1}{\lambda_1} + \lambda_2 c_{11} = 0 \quad (5)$$

From Eq4 and Eq5 above,

$$\frac{1}{\lambda_1} \stackrel{(5)}{=} c_{1\mu} + \lambda_2 c_{11}$$

$$\begin{aligned} \tilde{\mu} &\stackrel{(4)}{=} \lambda_1 (c_{\mu\mu} + \lambda_2 c_{1\mu}) \\ &= \frac{c_{\mu\mu} + \lambda_2 c_{1\mu}}{c_{1\mu} + \lambda_2 c_{11}} \end{aligned}$$

where

- $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$
- $c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \mathbf{1}$
- $c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$

(f) From Eq1, consider:

$$\mathbf{w}_*^T \times (1) : \mathbf{w}_*^T \boldsymbol{\mu} - \frac{1}{\lambda_1} \mathbf{w}_*^T \Sigma \mathbf{w}_* + \lambda_2 \mathbf{w}_*^T \mathbf{1} \stackrel{(2,3)}{=} \tilde{u} - \frac{1}{\lambda_1} \tilde{\sigma}^2 + \lambda_2 = 0$$

$$\begin{aligned} \implies \quad \tilde{\sigma}^2 &= \lambda_1 (\tilde{\mu} + \lambda_2) \\ &= \frac{\tilde{\mu} + \lambda_2}{c_{1\mu} + \lambda_2 c_{11}} \\ &= \frac{c_{\mu\mu} + 2\lambda_2 c_{1\mu} + \lambda_2^2 c_{11}}{(c_{1\mu} + \lambda_2 c_{11})^2} \quad (\text{By substituting } \tilde{\mu}) \end{aligned}$$

(g) From (f),

$$\tilde{\sigma}^2 (c_{1\mu} + \lambda_2 c_{11})^2 = c_{\mu\mu} + 2\lambda_2 c_{1\mu} + \lambda_2^2 c_{11}$$

Expanding the square and rearranging the terms, we can get the following quadratic equation:

$$c_{11}(c_{11}\tilde{\sigma}^2 - 1)\lambda_2^2 + 2c_{1\mu}(c_{11}\tilde{\sigma}^2 - 1)\lambda_2 + (c_{1\mu}^2\tilde{\sigma}^2 - c_{\mu\mu}) = 0,$$

which can be used to solve for λ_2 .

(h) **Approach 1:**

From (e) we have

$$\tilde{\mu} = \frac{c_{\mu\mu} + \lambda_2 c_{1\mu}}{c_{1\mu} + \lambda_2 c_{11}} \quad \implies \quad \lambda_2 = \frac{c_{\mu\mu} - \tilde{\mu} c_{1\mu}}{\tilde{\mu} c_{11} - c_{1\mu}}. \quad (6)$$

From (f) we have

$$\begin{aligned}
\tilde{\sigma}^2 &= \frac{c_{\mu\mu} + 2\lambda_2 c_{1\mu} + \lambda_2^2 c_{11}}{(c_{1\mu} + \lambda_2 c_{11})^2} \\
&= \frac{c_{\mu\mu} + 2 \left(\frac{c_{\mu\mu} - \tilde{\mu} c_{1\mu}}{\tilde{\mu} c_{11} - c_{1\mu}} \right) c_{1\mu} + \left(\frac{c_{\mu\mu} - \tilde{\mu} c_{1\mu}}{\tilde{\mu} c_{11} - c_{1\mu}} \right)^2 c_{11}}{(c_{1\mu} + \left(\frac{c_{\mu\mu} - \tilde{\mu} c_{1\mu}}{\tilde{\mu} c_{11} - c_{1\mu}} \right) c_{11})^2} \quad (\text{by substituting (6)}) \\
&= \frac{[c_{\mu\mu}(\tilde{\mu} c_{11} - c_{1\mu})^2 + 2(c_{\mu\mu} - \tilde{\mu} c_{1\mu})c_{1\mu}(\tilde{\mu} c_{11} - c_{1\mu}) + (c_{\mu\mu} - \tilde{\mu} c_{1\mu})^2 c_{11}]/(\tilde{\mu} c_{11} - c_{1\mu})^2}{[c_{1\mu}(\tilde{\mu} c_{11} - c_{1\mu}) + (c_{\mu\mu} - \tilde{\mu} c_{1\mu})c_{11}]^2/(\tilde{\mu} c_{11} - c_{1\mu})^2} \\
&= \frac{(c_{11}c_{\mu\mu} - c_{1\mu}^2)(c_{11}\tilde{\mu}^2 - 2c_{1\mu}\tilde{\mu} + c_{\mu\mu})}{(c_{11}c_{\mu\mu} - c_{1\mu}^2)^2} \\
&= \frac{c_{11}\tilde{\mu}^2 - 2c_{1\mu}\tilde{\mu} + c_{\mu\mu}}{c_{11}c_{\mu\mu} - c_{1\mu}^2}.
\end{aligned}$$

The result here is the same as that in (b).

Approach 2:

Consider

$$\underbrace{c_{11}(c_{11}\tilde{\sigma}^2 - 1)}_a \lambda_2^2 + \underbrace{2c_{1\mu}(c_{11}\tilde{\sigma}^2 - 1)}_b \lambda_2 + \underbrace{(c_{1\mu}^2\tilde{\sigma}^2 - c_{\mu\mu})}_c = 0$$

Using quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain

$$\lambda_2 = \frac{-2c_{1\mu}(c_{11}\tilde{\sigma}^2 - 1) \pm \sqrt{4\Delta(c_{11}\tilde{\sigma}^2 - 1)}}{2c_{11}(c_{11}\tilde{\sigma}^2 - 1)} = -\frac{c_{1\mu}}{c_{11}} \pm \frac{1}{c_{11}} \sqrt{\frac{\Delta}{c_{11}\tilde{\sigma}^2 - 1}}$$

where $\Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2$.

From results in (e):

$$\begin{aligned}
\tilde{\mu} &= \frac{c_{\mu\mu} + \lambda_2 c_{1\mu}}{c_{1\mu} + \lambda_2 c_{11}} \\
&= \frac{c_{\mu\mu} - \frac{c_{1\mu}^2}{c_{11}} \pm \frac{c_{1\mu}}{c_{11}} \sqrt{\frac{\Delta}{c_{11}\tilde{\sigma}^2 - 1}}}{\pm \sqrt{\frac{\Delta}{c_{11}\tilde{\sigma}^2 - 1}}} \\
\Rightarrow \tilde{\mu} - \frac{c_{1\mu}}{c_{11}} &= \pm \frac{\Delta}{c_{11}} \sqrt{\frac{c_{11}\tilde{\sigma}^2 - 1}{\Delta}} = \pm \frac{\sqrt{\Delta(c_{11}\tilde{\sigma}^2 - 1)}}{c_{11}}. \quad (7)
\end{aligned}$$

Using results from (f), Eq7, and consider $\lambda_2 = -\frac{c_{1\mu}}{c_{11}} + \frac{1}{c_{11}}\sqrt{\frac{\Delta}{c_{11}\tilde{\sigma}^2-1}}$,

$$\begin{aligned}
\tilde{\sigma}^2 &= \frac{c_{\mu\mu} + 2\lambda_2 c_{1\mu} + \lambda_2^2 c_{11}}{(c_{1\mu} + \lambda_2 c_{11})^2} \\
&= \left[c_{\mu\mu} + 2\left(-\frac{c_{1\mu}}{c_{11}} + \frac{1}{c_{11}}\sqrt{\frac{\Delta}{c_{11}\tilde{\sigma}^2-1}}\right)c_{1\mu} + \left(-\frac{c_{1\mu}}{c_{11}} + \frac{1}{c_{11}}\sqrt{\frac{\Delta}{c_{11}\tilde{\sigma}^2-1}}\right)^2 c_{11} \right] \frac{c_{11}\tilde{\sigma}^2-1}{\Delta} \\
&= \left(\frac{\Delta}{c_{11}(c_{11}\tilde{\sigma}^2-1)} + c_{\mu\mu} - \frac{c_{1\mu}^2}{c_{11}} \right) \frac{c_{11}\tilde{\sigma}^2-1}{\Delta} \\
&= \Delta^{-1} \left(\frac{c_{11}\Delta(c_{11}\tilde{\sigma}^2-1)}{c_{11}^2} + \frac{\Delta}{c_{11}} \right) \\
&\stackrel{(7)}{=} \Delta^{-1} \left[c_{11} \left(\tilde{\mu} - \frac{c_{1\mu}}{c_{11}} \right)^2 + c_{\mu\mu} - \frac{c_{1\mu}^2}{c_{11}} \right],
\end{aligned}$$

which shows that the relationship between $\tilde{\sigma}^2$ and $\tilde{\mu}$ is the same as the relationship between $\sigma^2(\boldsymbol{\mu}^*)$ and $\boldsymbol{\mu}^*$ in (b).