

Contents

1	Question 1 10 marks	2
1.1	Question 1(a) 1 mark	2
1.2	Question 1(b) 3 marks	3
1.3	Question 1(c) 2 marks	4
1.4	Question 1(d) 2 marks	5
1.5	Question 1(e) 2 marks	5
2	Question 2 40 marks	6
2.1	Question 2(a) 10 marks	6
2.2	Question 2(b) 14 marks	7
2.2.1	Question 2(b) 14 marks Diagram	8
2.3	Question 2(c) 6 marks	9
2.3.1	Question 2(c) 6 marks Diagram	10
2.4	Question 2(d) 3 marks	11
2.5	Question 2(e) 5 marks	12
2.6	Question 2(f) 2 marks	13
3	Question 3 20 marks	14
3.1	Question 3(a) 2 marks	14
3.2	Question 3(b) 3 marks	14
3.3	Question 3(c) 3 marks	15
3.4	Question 3(d) 4 marks	15
3.4.1	Question 3(d) 4 marks Diagram	16
3.5	Question 3(e) 4 marks	17
3.6	Question 3(f) 4 marks	18
4	Question 4 22 marks	19
4.1	Question 4(a) 3 marks	19
4.2	Question 4(b) 6 marks	20
4.3	Question 4(c) 6 marks	21
4.4	Question 4(d) 2 marks	22
4.5	Question 4(e) 5 marks	23
5	Question 5 8 marks	24
5.1	Question 5(a) 4 marks	24
5.2	Question 5(b) 4 marks	25

© All rights reserved by Kim HUNG

1 Question 1 | 10 marks

1.1 Question 1(a) | 1 mark

Concepts (Question 1(a) | 1 mark)

...

Solution: Refer to the following code.

```
1 n <- length(VA) / 10
2 ##### OR #####
3 length(VA) / 10
```

1.2 Question 1(b) | 3 marks

Concepts (Question 1(b) | 3 marks)

...

Solution: Refer to the following code.

```

1 PA <- colMeans(matrix(VA, nrow = 10))
2 PB <- colMeans(matrix(VB, nrow = 10))
3 PC <- colMeans(matrix(VC, nrow = 10))
4 ##### OR #####
5 PA <- rowMeans(matrix(VA, ncol = 10, byrow = T))
6 PB <- rowMeans(matrix(VB, ncol = 10, byrow = T))
7 PC <- rowMeans(matrix(VC, ncol = 10, byrow = T))
8 ##### OR #####
9 n <- length(VA)/10
10 PA <- rep(NA, n)
11 PB <- rep(NA, n)
12 PC <- rep(NA, n)
13 for (i in 1:n){
14   PA[i] <- mean(VA[(10*i - 9):(10*i)])
15   PB[i] <- mean(VB[(10*i - 9):(10*i)])
16   PC[i] <- mean(VC[(10*i - 9):(10*i)])
17 }
18 ##### OR #####
19 n <- length(VA)/10
20 PA <- matrix(VA, nrow = n, byrow = T)
21 PB <- matrix(VB, nrow = n, byrow = T)
22 PC <- matrix(VC, nrow = n, byrow = T)
23 PA <- apply(PA, 1, mean)
24 PB <- apply(PB, 1, mean)
25 PC <- apply(PC, 1, mean)
26 ##### OR #####
27 n <- length(VA)/10
28 PA <- matrix(VA, nrow = n)
29 PB <- matrix(VB, nrow = n)
30 PC <- matrix(VC, nrow = n)
31 PA <- apply(PA, 2, mean)
32 PB <- apply(PB, 2, mean)
33 PC <- apply(PC, 2, mean)
  
```

Remark (Question 1(b) | 3 marks)

Refer to the following code.

```

1 # apply("a vector (in matrix form, e.g., R^{3 x 1})", "1 or 2", mean) will always give you a real number.
2 # and obviously, what you want for (PA/PB/PC) belongs to a n-dimensional vector, not a real number.
3 # E.g., apply(matrix(VA), 1, mean) -> real number.
4 #####
5 # apply("a vector (not a matrix form, e.g., R^3, c(1,2,3))", "1 or 2", mean) will always produce the error
6 # "Error in apply(...): dim(...) must have a postive length".
7 # E.g., apply(VA, 1, mean) -> ERROR.
  
```

1.3 Question 1(c) | 2 marks

Concepts (Question 1(c) | 2 marks)

...

Solution: Refer to the following code.

```
1 LA <- diff(log(PA))
2 LB <- diff(log(PB))
3 LC <- diff(log(PC))
4 ##### OR #####
5 n <- length(VA)/10
6 LA <- log(PA[-1] / PA[-n])
7 LB <- log(PB[-1] / PB[-n])
8 LC <- log(PC[-1] / PC[-n])
9 ##### OR #####
10 n <- length(VA)/10
11 LA <- log(PA[-1]) - log(PA[-n])
12 LB <- log(PB[-1]) - log(PB[-n])
13 LC <- log(PC[-1]) - log(PC[-n])
14 ##### OR #####
15 n <- length(VA)/10
16 LA <- log(PA[2:n] / PA[1:n-1])
17 LB <- log(PB[2:n] / PB[1:n-1])
18 LC <- log(PC[2:n] / PC[1:n-1])
19 ##### OR #####
20 n <- length(VA)/10
21 LA <- log(PA[2:n]) - log(PA[1:n-1])
22 LB <- log(PB[2:n]) - log(PB[1:n-1])
23 LC <- log(PC[2:n]) - log(PC[1:n-1])
24 ##### OR #####
25 n <- length(VA)/10
26 LA <- rep(NA, n-1)
27 LB <- rep(NA, n-1)
28 LC <- rep(NA, n-1)
29 for (i in 1:(n-1)){
30   LA[i] = log(PA[i+1]) - log(PA[i])
31   LB[i] = log(PB[i+1]) - log(PB[i])
32   LC[i] = log(PC[i+1]) - log(PC[i])
33 }
34 ##### OR #####
35 n <- length(VA)/10
36 LA <- rep(NA, n - 1)
37 LB <- rep(NA, n - 1)
38 LC <- rep(NA, n - 1)
39 for (i in 1:(n-1)){
40   LA[i] = log(PA[i+1] / PA[i])
41   LB[i] = log(PB[i+1] / PB[i])
42   LC[i] = log(PC[i+1] / PC[i])
43 }
```

Remark (Question 1(c) | 2 marks)

Refer to the followig code.

```
1 # First: PA[1:n-1] (They are the SAME.) PA[1:(n-1)]
2 # Second: for (i in 1:(n-1)) (They are NOT the SAME.) for (i in 1:n-1)
```

1.4 Question 1(d) | 2 marks

Concepts (Question 1(d) | 2 marks)

...

Solution: Refer to the following code.

```
1 ELA <- mean(LA)
2 ##### OR #####
3 mean(LA)
```

1.5 Question 1(e) | 2 marks

Concepts (Question 1(e) | 2 marks)

...

Solution: Refer to the following code.

```
1 S <- cbind(LA, LB, LC)
2 var(S)
3 ##### OR #####
4 S <- cbind(LA, LB, LC)
5 cov(S)
6 ##### OR #####
7 var(cbind(LA, LB, LC))
8 ##### OR #####
9 cov(cbind(LA, LB, LC))
```

2 Question 2 | 40 marks

2.1 Question 2(a) | 10 marks

Concepts (Question 2(a) | 10 marks)

...

Solution: Compute parameters:

$$c_{11} = \mathbf{1}^T \hat{\Sigma}^{-1} \mathbf{1} = 0.583$$

$$c_{1\mu} = \mathbf{1}^T \hat{\Sigma}^{-1} \hat{\mu} = 0.0354$$

$$c_{\mu\mu} = \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu} = 0.0024636$$

$$\Delta = c_{11}c_{\mu\mu} - c_{1\mu}^2 = 0.0001831188$$

Set the target expected return as $\mu_* = 0.08$, the variance-minimized portfolio is

$$\begin{aligned} w_*(\mu_*) &= \Delta^{-1} \left(c_{11} \hat{\Sigma}^{-1} \hat{\mu} - c_{1\mu} \hat{\Sigma}^{-1} \mathbf{1} \right) \mu_* + \Delta^{-1} \left(c_{\mu\mu} \hat{\Sigma}^{-1} \mathbf{1} - c_{1\mu} \hat{\Sigma}^{-1} \hat{\mu} \right) \\ &= \begin{pmatrix} -0.2125 \\ 0.6374 \\ 0.5751 \end{pmatrix} \end{aligned}$$

The volatility is

$$\sigma_*(\mu_*) = \sqrt{\Delta^{-1} (c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu})} = 1.703$$

2.2 Question 2(b) | 14 marks

Concepts (Question 2(b) | 14 marks)

...

Solution: Note that:

$$\mathbf{w}_{\minvar} = \frac{\hat{\Sigma}^{-1}\mathbf{1}}{c_{11}} = \begin{pmatrix} 0.1801 \\ 0.4237 \\ 0.3962 \end{pmatrix}$$

The expected return and volatility of the overall MVP can be found by

$$\mu_{\minvar} = \frac{c_{1\mu}}{c_{11}} = 0.0607, \quad \text{and} \quad \sigma_{\minvar} = \sqrt{\frac{1}{c_{11}}} = 1.3097$$

The efficient frontier is the upper part of the minimum-variance set, that is:

$$\begin{aligned} \sigma_*^2(\mu_*) &= \Delta^{-1} \left[c_{11} \left(\mu_* - \frac{c_{1\mu}}{c_{11}} \right)^2 + c_{\mu\mu} - \frac{c_{1\mu}^2}{c_{11}} \right] \\ &= 3183.726 (\mu_* - 0.0607)^2 + 1.7153, \quad \mu_* \geq \mu_{\minvar} = 0.0607 \end{aligned}$$

OR

$$\begin{aligned} \sigma_*^2(\mu_*) &= \Delta^{-1} (c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu}) \\ &= \frac{c_{11}}{\Delta} \mu_*^2 - \frac{2c_{1\mu}}{\Delta} \mu_* + \frac{c_{\mu\mu}}{\Delta} \\ &= 3183.726\mu_*^2 - 386.6343\mu_* + 13.4536, \quad \mu_* \geq \mu_{\minvar} = 0.0607 \end{aligned}$$

Note that:

$$\begin{aligned} \mu_a &= 0.02, \sigma_a = \sqrt{7} \approx 2.645751 \\ \mu_b &= 0.06, \sigma_b = \sqrt{6} \approx 2.449490 \\ \mu_c &= 0.08, \sigma_c = \sqrt{8} \approx 2.828427 \end{aligned}$$

Hence, in the plot, $\mu_a < \mu_b < \mu_c$. However, $\sigma_b < \sigma_a < \sigma_c$.

Remark (Question 2(b) | 14 marks)

In the plot, $\mu_a < \mu_b < \mu_c$; $\sigma_b < \sigma_a < \sigma_c$.

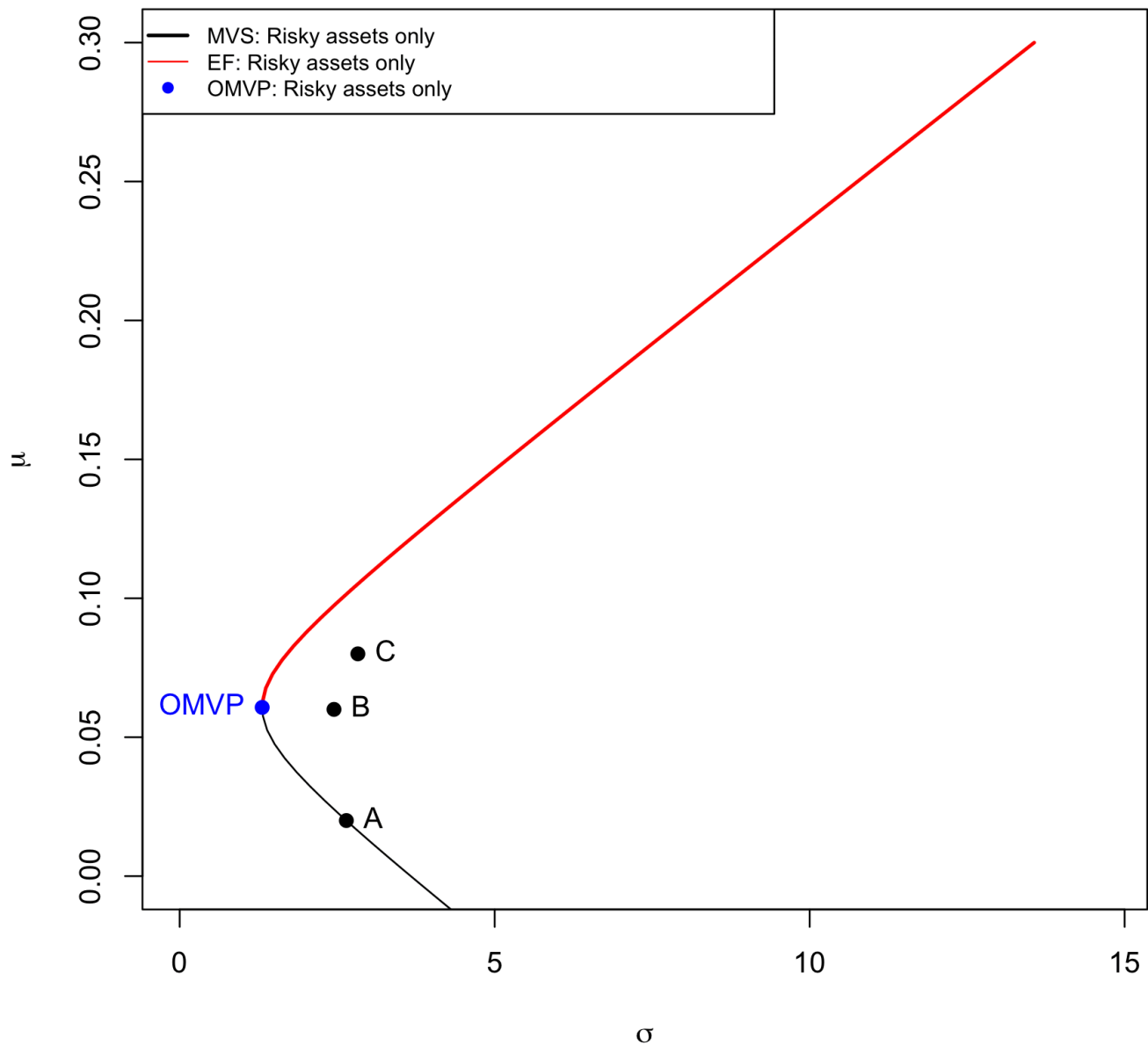
i.e., from the bottom to top: $\begin{pmatrix} C \\ \uparrow \\ B \\ \uparrow \\ A \end{pmatrix}$; from the left to right: $(B \rightarrow A \rightarrow C)$.

Remark (Question 2(b) | 14 marks)

(σ_a, μ_a) is on the MVS.

2.2.1 Question 2(b) | 14 marks | Diagram

MVS & EF & OMVP (with Risky Assets only)



2.3 Question 2(c) | 6 marks

Concepts (Question 2(c) | 6 marks)

...

Solution: The tangency portfolio is given by

$$w_M = \frac{\hat{\Sigma}^{-1}(\hat{\mu} - r\mathbf{1})}{c_{1\mu} - rc_{11}} = \begin{pmatrix} -0.8432 \\ 0.9808 \\ 0.8624 \end{pmatrix}$$

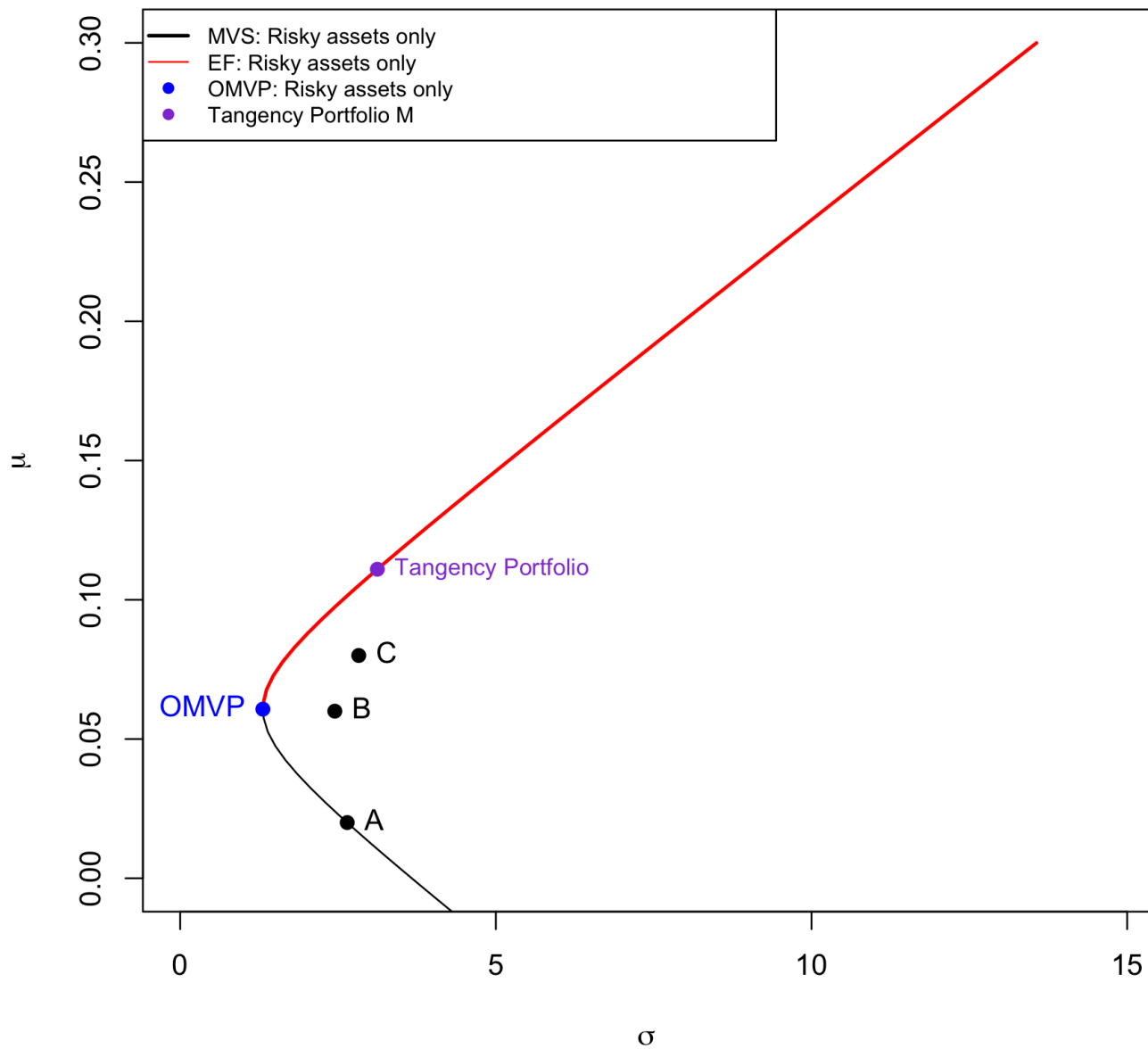
$$\text{Expected return: } \mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}} = 0.111$$

$$\text{Volatility: } \sigma_M = \frac{|\gamma|}{c_{1\mu} - rc_{11}} = 3.1235,$$

$$\text{where } \gamma = \sqrt{(\hat{\mu} - r\mathbf{1})^T \hat{\Sigma}^{-1} (\hat{\mu} - r\mathbf{1})} = 0.0195.$$

2.3.1 Question 2(c) | 6 marks | Diagram

MVS & EF & OMVP (with Risky Assets only)



2.4 Question 2(d) | 3 marks

Concepts (Question 2(d) | 3 marks)

...

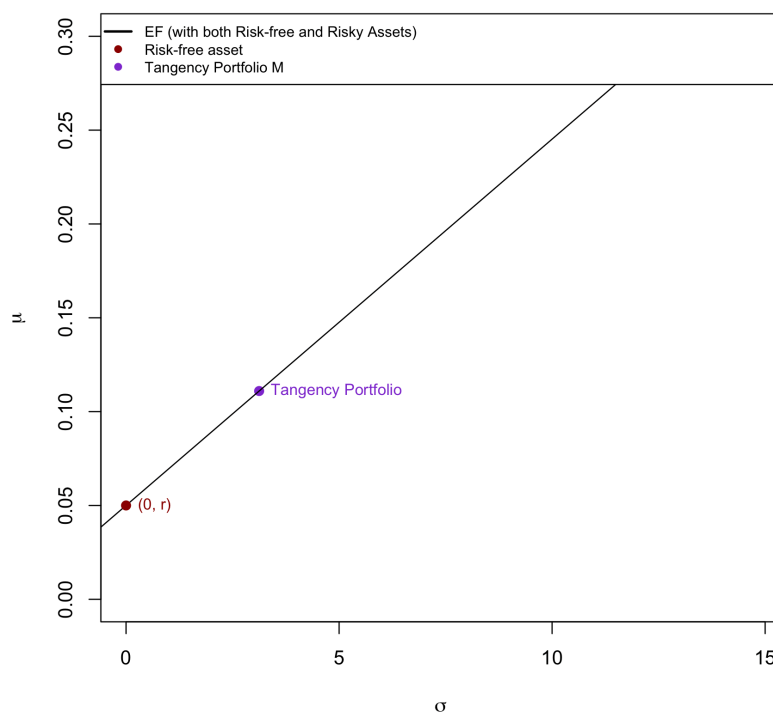
Solution: The efficient frontier is the line:

$$\mu_* = |\gamma|\sigma_* + r = 0.0195\sigma_* + 0.05$$

Remark (Question 2(d) | 3 marks)

- The question require you to draw the efficient frontier constructed from both the risk free asset and the three stocks.
 - i.e., it did not force you to indicate the three stocks.
 - i.e., it is not a must to plot the three stocks on the efficient frontier.
 - Once you show the correct efficient frontier (implicitly, constructed from both risk free asset and the three stocks), you will be fine.
- | For example, if the line passes through the $(0, r)$ and the correct tangency portfolio, the line has to be the correct efficient frontier constructed from both risk free asset and the three stocks.

EF (with both Risk-free and Risky Assets)



2.5 Question 2(e) | 5 marks

Concepts (Question 2(e) | 5 marks)

...

Solution: Refer to the following code.

```
1 library(quadprog)
2 n = 3
3 muStar = 0.08
4 V1 = matrix(1,n,1)
5 V0 = matrix(0,n,1)
6 Amat = cbind(M,V1,diag(n))
7 b0 = matrix(c(muStar,1,V0),n+2,1)
8 p = solve.QP(C, V0, Amat, b0, 2)
9 W0 = p$solution
10 #####
11 ##### OR #####
12 ##### b0 without 1 is fine #####
13 #####
14 library(quadprog)
15 n = 3
16 muStar = 0.08
17 V1 = matrix(1,n,1)
18 V0 = matrix(0,n,1)
19 Amat = cbind(M,V1,diag(n))
20 b0 = matrix(c(muStar,1,V0),n+2)
21 p = solve.QP(C, V0, Amat, b0, 2)
22 W0 = p$solution
```

Remark (Question 2(e) | 5 marks)

All undefined variable –1 (max).

- The "M" and "C" was defined in the question.

2.6 Question 2(f) | 2 marks

Concepts (Question 2(f) | 2 marks)

...

Solution: For a portfolio on the efficient frontier positioned at the top right of the tangency portfolio, the weight allocated to the risk-free asset is negative. This indicates borrowing at the risk-free rate to invest more in risky assets.

3 Question 3 | 20 marks

3.1 Question 3(a) | 2 marks

Concepts (Question 3(a) | 2 marks)

...

Solution:

$$\begin{aligned}0.8 + 0.3 + x &= 1 \\ x &= -0.1\end{aligned}$$

3.2 Question 3(b) | 3 marks

Concepts (Question 3(b) | 3 marks)

...

Solution:

$$\text{Cov}(R_p, R_q) = \mathbf{w}_p^T \hat{\Sigma} \mathbf{w}_q = 4.9$$

3.3 Question 3(c) | 3 marks

Concepts (Question 3(c) | 3 marks)

...

Solution:

$$\beta_p = \frac{\sigma_{pM}}{\sigma_M^2} = \frac{w_p^T \hat{\Sigma} w_M}{\sigma_M^2} = -0.2624$$

.

3.4 Question 3(d) | 4 marks

Concepts (Question 3(d) | 4 marks)

...

Solution: The security market line: $\mu = (\mu_M - r)\beta + r = 0.061\beta + 0.05$. And the beta for other assets and portfolios are:

$$\beta_q = \frac{w_q^T \hat{\Sigma} w_M}{\sigma_M^2} = -0.3935, \quad \beta_a = \frac{w_1^T \hat{\Sigma} w_M}{\sigma_M^2} = -0.4923$$

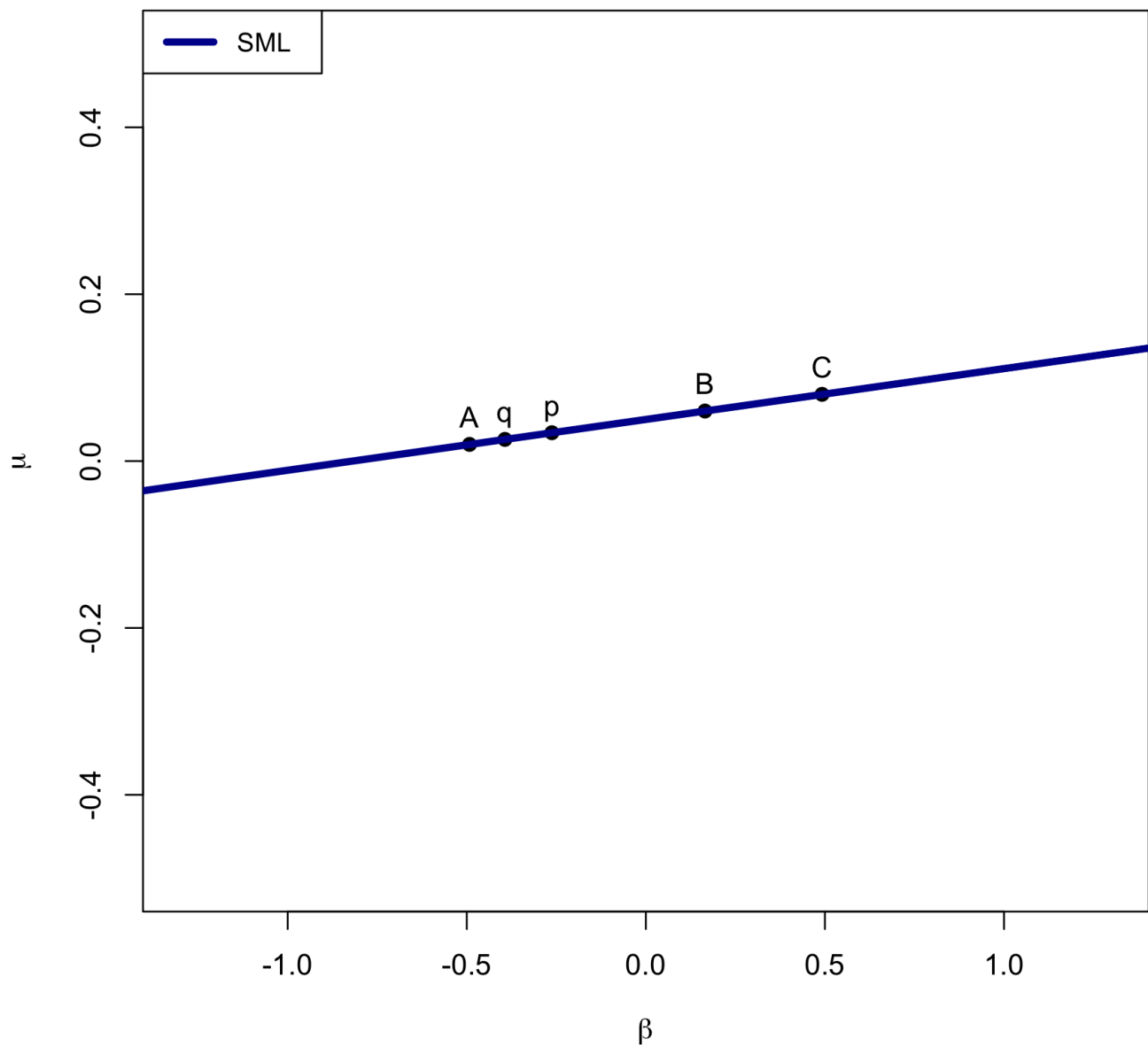
$$\beta_b = \frac{w_2^T \hat{\Sigma} w_M}{\sigma_M^2} = 0.1651, \quad \beta_c = \frac{w_3^T \hat{\Sigma} w_M}{\sigma_M^2} = 0.492$$

where w_i is a vector with 1 in i^{th} index and 0 elsewhere. Note that

$$\mu_p = w_p^T \mu = 0.034, \quad \mu_q = w_q^T \mu = 0.026$$

3.4.1 Question 3(d) | 4 marks | Diagram

SML



3.5 Question 3(e) | 4 marks

Concepts (Question 3(e) | 4 marks)

...

Solution:

$$\sigma_p = \sqrt{w_p^T \hat{\Sigma} w_p} = \sqrt{4.05} = 2.0125$$

Sharpe ratio:

$$\frac{\mu_p - r}{\sigma_p} = -0.008$$

Treynor ratio:

$$\frac{\mu_p - r}{\beta_p} = 0.061$$

3.6 Question 3(f) | 4 marks

Concepts (Question 3(f) | 4 marks)

...

Solution: Note that:

$$w_p = \begin{pmatrix} 0.7 \\ 0.2 \\ 0.1 \end{pmatrix}, \quad p = \begin{pmatrix} 34 \\ 78 \\ 120 \end{pmatrix}, \quad n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ 200 \end{pmatrix} \Rightarrow 0.1 = \frac{(120)(200)}{34n_1 + 78n_2 + 120n_3} = \frac{(120)(200)}{P_0}$$

where p, n and P_0 are the initial price of the three assets, the initial holding of Portfolio w_p in three assets, and the initial wealth of the Portfolio w_p , respectively.

Hence, we get:

$$P_0 = \$240000.$$

Note that:

$$\beta_p = -0.2624, \quad P_0 = \$240000, \quad M_0 = \$200, \quad N = \beta_p \frac{P_0}{M_0} = -314.878$$

Therefore, to hedge the risk of one unit of w_p , the investor should take a short position on -314.878 units of the future contract of market portfolio.

i.e., to hedge the risk of one unit of w_p , the investor should take a long position on 314.878 units of the future contract of market portfolio.

4 Question 4 | 22 marks

4.1 Question 4(a) | 3 marks

Concepts (Question 4(a) | 3 marks)

...

Solution: The expected return and volatility of this portfolio are

$$\mu = w^T \boldsymbol{\mu} \quad \text{and} \quad \sigma = \sqrt{w^T \Sigma w}$$

Hence, the slope of the line from (σ, μ) and $(0, r)$ is

$$\frac{\mu - r}{\sigma - 0} = \frac{w^T \boldsymbol{\mu} - r w^T \mathbf{1}}{\sqrt{w^T \Sigma w}} = \frac{w^T (\boldsymbol{\mu} - r \mathbf{1})}{\sqrt{w^T \Sigma w}}$$

4.2 Question 4(b) | 6 marks

Concepts (Question 4(b) | 6 marks)

...

Solution: Take differentiation on the slope with respect to w

$$\begin{aligned} \frac{d}{dw} \frac{w^T(\mu - r\mathbf{1})}{\sqrt{w^T \Sigma w}} &= \frac{(\mu - r\mathbf{1})(w^T \Sigma w)^{1/2} - (w^T \Sigma w)^{-1/2}(\Sigma w)w^T(\mu - r\mathbf{1})}{w^T \Sigma w} = 0 \\ \Rightarrow (w^T \Sigma w)^{-1/2}(\Sigma w)w^T(\mu - r\mathbf{1}) &= (\mu - r\mathbf{1})(w^T \Sigma w)^{1/2} \\ \Rightarrow (\Sigma w)w^T(\mu - r\mathbf{1}) &= (\mu - r\mathbf{1})(w^T \Sigma w) \\ \Rightarrow ww^T(\mu - r\mathbf{1}) &= (w^T \Sigma w)\Sigma^{-1}(\mu - r\mathbf{1}) \\ \Rightarrow w &= \frac{w^T \Sigma w}{w^T(\mu - r\mathbf{1})}\Sigma^{-1}(\mu - r\mathbf{1}) \end{aligned}$$

Hence, consider $w = w_M$, $\frac{w^T \Sigma w}{w^T(\mu - r\mathbf{1})} = \frac{1}{\lambda} \in \mathbb{R}$, we get:

$$w_M = \frac{1}{\lambda}\Sigma^{-1}(\mu - r\mathbf{1}) \Rightarrow \mu - r\mathbf{1} = \lambda \Sigma w_M$$

4.3 Question 4(c) | 6 marks

Concepts (Question 4(c) | 6 marks)

...

Solution:

$$\boldsymbol{\mu} - r\mathbf{1} = \lambda \boldsymbol{\Sigma} \mathbf{w}_M$$

$$\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - r \boldsymbol{\Sigma}^{-1} \mathbf{1} = \lambda \mathbf{w}_M$$

$$\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} - r \mathbf{1}^T \boldsymbol{\Sigma}^{-1} = \lambda \mathbf{w}_M^T$$

$$\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1} - r \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1} = \lambda \mathbf{w}_M^T \mathbf{1} = \lambda$$

Therefore,

$$c_{1\mu} = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1} \quad \text{and} \quad c_{11} = \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}.$$

Hence,

$$\lambda = c_{1\mu} - r c_{11}$$

$$\mathbf{w}_M = \frac{1}{\lambda} \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r \mathbf{1}) = \frac{\boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r \mathbf{1})}{c_{1\mu} - r c_{11}}$$

4.4 Question 4(d) | 2 marks

Concepts (Question 4(d) | 2 marks)

...

Solution:

$$\begin{aligned}\mu - r\mathbf{1} &= \lambda \Sigma \mathbf{w}_M \\ \mathbf{w}_M^T \mu - r \mathbf{w}_M^T \mathbf{1} &= \lambda \mathbf{w}_M^T \Sigma \mathbf{w}_M \\ \mu_M - r &= \lambda \sigma_M^2\end{aligned}$$

4.5 Question 4(e) | 5 marks

Concepts (Question 4(e) | 5 marks)

...

Solution:

$$\begin{aligned}\frac{\mu_M - r}{\sigma_M^2} &= \frac{\mathbf{w}_M^T (\boldsymbol{\mu} - r\mathbf{1})}{\mathbf{w}_M^T \boldsymbol{\Sigma} \mathbf{w}_M} = \lambda \in \mathbb{R} \\ \Rightarrow \boldsymbol{\mu} - r\mathbf{1} &= \lambda \boldsymbol{\Sigma} \mathbf{w}_M = \boldsymbol{\Sigma} \mathbf{w}_M \lambda = \boldsymbol{\Sigma} \mathbf{w}_M \frac{\mu_M - r}{\sigma_M^2} \\ \Rightarrow \mathbf{w}_a^T \boldsymbol{\mu} - r \mathbf{w}_a^T \mathbf{1} &= \mathbf{w}_a^T \boldsymbol{\Sigma} \mathbf{w}_M \frac{\mu_M - r}{\sigma_M^2} \\ \Rightarrow E(R_a) - r &= \text{Cov}(R_a, R_M) \frac{\mu_M - r}{\sigma_M^2} = (\mu_M - r) \frac{\text{Cov}(R_a, R_M)}{\sigma_M^2} = (\mu_M - r) \beta_a\end{aligned}$$

Hence, we get:

$$\mu_a = r + (\mu_M - r) \beta_a \quad \text{or} \quad E(R_a) = r + (E(R_M) - r) \beta_a,$$

which is the CAPM equation.

5 Question 5 | 8 marks

5.1 Question 5(a) | 4 marks

Concepts (Question 5(a) | 4 marks)

...

Solution: Consider $E(P_\alpha) = \mu_P$, $Var(P_\alpha) = \sigma_P^2$, we get:

$$\mu_P = \alpha r + (1 - \alpha)\mu_a, \quad \sigma_P^2 = (1 - \alpha)^2 \sigma_a^2 \Rightarrow 1 - \alpha = \pm \frac{\sigma_P}{\sigma_a}$$

Note that: $0 < \alpha < 1 \Rightarrow 1 > 1 - \alpha > 0$, $\sigma_P > 0$, and $\sigma_a > 0 \Rightarrow 1 - \alpha = \frac{\sigma_P}{\sigma_a}$, $\alpha = 1 - \frac{\sigma_P}{\sigma_a}$.

Hence, by using $\mu_P = \alpha r + (1 - \alpha)\mu_a$, we get:

$$\begin{aligned} \mu_P &= \left(1 - \frac{\sigma_P}{\sigma_a}\right) r + \frac{\sigma_P}{\sigma_a} \mu_a \\ &= r - \frac{r}{\sigma_a} \sigma_P + \frac{\mu_a}{\sigma_a} \sigma_P \\ &= \left(\frac{\mu_a - r}{\sigma_a}\right) \sigma_P + r. \end{aligned}$$

Hence, we get: $\mu_P = \left(\frac{\mu_a - r}{\sigma_a}\right) \sigma_P + r$.

Remark (Question 5(b) | 4 marks)

- $\sigma > 0$ if the asset is a risky asset.
- Rejected case: $1 - \alpha = -\frac{\sigma_P}{\sigma_a}$.

5.2 Question 5(b) | 4 marks

Concepts (Question 5(b) | 4 marks)

...

Solution: Consider $E(Q_\alpha) = \mu_Q$, $Var(Q_\alpha) = \sigma_Q^2$, we get:

$$\mu_Q = \alpha\mu_b + (1 - \alpha)\mu_a, \quad \sigma_Q^2 = \alpha^2\sigma_b^2 + (1 - \alpha)^2\sigma_a^2 + 2\alpha(1 - \alpha)\sigma_{ab}$$

Consider

$$\begin{aligned} \sigma_Q^2 &= \alpha^2\sigma_b^2 + (1 - \alpha)^2\sigma_a^2 + 2\alpha(1 - \alpha)\sigma_{ab} \\ &= \alpha^2\sigma_b^2 + \sigma_a^2 - 2\alpha\sigma_a^2 + \alpha^2\sigma_a^2 + 2\alpha\sigma_{ab} - 2\alpha^2\sigma_{ab} \\ &= (\sigma_a^2 - 2\sigma_{ab} + \sigma_b^2)\alpha^2 + 2(\sigma_{ab} - \sigma_a^2)\alpha + \sigma_a^2 \end{aligned}$$

Note that the statement requires linearity for both μ_Q and σ_Q , i.e., $\sigma_Q = u\alpha + v$, where u is a non-zero constant and v is a constant. Since μ_Q is already linear in α , we focus on σ_Q . Note that $\sigma_Q = u\alpha + v$ implies the discriminant (of the quadratic in α) = 0.

$$\begin{aligned} 4(\sigma_{ab} - \sigma_a^2)^2 - 4(\sigma_a^2 - 2\sigma_{ab} + \sigma_b^2)(\sigma_a^2) &= 0 \\ (\sigma_{ab} - \sigma_a^2)^2 &= (\sigma_a^2 - 2\sigma_{ab} + \sigma_b^2)(\sigma_a^2) \\ \sigma_{ab}^2 - 2\sigma_{ab}\sigma_a^2 + \sigma_a^4 &= \sigma_a^4 - 2\sigma_{ab}\sigma_a^2 + \sigma_a^2\sigma_b^2 \\ \sigma_{ab}^2 &= \sigma_a^2\sigma_b^2 \end{aligned}$$

Note that $\sigma_{ab} > 0$, $\sigma_a > 0$ and $\sigma_b > 0$, $\sigma_{ab} = \sigma_a\sigma_b$. Hence, the condition is:

$$\sigma_{ab}^2 = \sigma_a^2\sigma_b^2 \Rightarrow \sigma_{ab} = \sigma_a\sigma_b$$

such that

$$\sigma_Q^2 = (\alpha\sigma_b + (1 - \alpha)\sigma_a)^2 \Rightarrow \sigma_Q^2 = \alpha\sigma_b + (1 - \alpha)\sigma_a = (\sigma_b - \sigma_a)\alpha + \sigma_a = u\alpha + v$$

Remark (Question 5(b) | 4 marks)

- $\sigma > 0$ if the asset is a risky asset.
- Rejected case: $\sigma_{ab} = -\sigma_a\sigma_b$.
- Condition: $X := \sigma_a^2 + \sigma_b^2 - 2\sigma_{ab} = 0$ is WRONG.

| $X = 0$ ensure that σ_Q^2 (Variance of Q_α) is linear in α NOT σ_Q (Volatility of Q_α).