

Contents

1 Question 1 (14 marks)	2
1.1 Question 1(a) (6 marks)	2
1.2 Question 1(b) (2 marks)	2
1.3 Question 1(c) (3 marks)	2
1.4 Question 1(d) (3 marks)	2
2 Question 2 (39 marks)	3
2.1 Question 2(a) (10 marks)	3
2.2 Question 2(b) (10 marks)	4
2.3 Question 2(c) (4 marks)	5
2.4 Question 2(d) (5 marks)	6
2.5 Question 2(e) (5 marks)	6
2.6 Question 2(f) (5 marks)	7
3 Question 3 (17 marks)	8
3.1 Question 3(a) (4 marks)	8
3.2 Question 3(b) (4 marks)	8
3.3 Question 3(c) (2 marks)	8
3.4 Question 3(d) (3 marks)	8
3.5 Question 3(e) (4 marks)	8
4 Question 4 (24 marks)	9
4.1 Question 4(a) (4 marks)	9
4.2 Question 4(b) (2 marks)	9
4.3 Question 4(c) (6 marks)	9
4.4 Question 4(d) (5 marks)	10
4.5 Question 4(e) (5 marks)	11
4.6 Question 4(f) (2 marks)	12
5 Question 5 (6 marks)	13
5.1 Question 5(a) (2 marks)	13
5.2 Question 5(b) (2 marks)	13
5.3 Question 5(c) (2 marks)	13

© All rights reserved by Kim HUNG

1 Question 1 (14 marks)

1.1 Question 1(a) (6 marks)

```
1 xxx1 = "2019-07-21"    OR    "2019-7-21"  
2 xxx2 = "2025-08-31"    OR    "2025-8-31"  
3 xxx3 = "AdjClose"  
4 xxx4 = paste0  
5 xxx5 = ".HK"  
6 xxx6 = "d"
```

1.2 Question 1(b) (2 marks)

- *Data* and *n* are not defined.

1.3 Question 1(c) (3 marks)

- Defines *Data* and *n*.

1.4 Question 1(d) (3 marks)

```
1 nP <- dim(Data)[1]  
2  
3 logR <- log(as.matrix(Data[-1,]) / as.matrix(Data[-nP,]))  
4 logR_2 <- log(Data[-1,] / Data[-nP,])  
5 logR_3 <- diff(log(Data))  
6 logR_4 <- diff(as.matrix(log(Data)))  
7  
8 mu.hat <- apply(logR,2,mean)  
9 mu.hat_2 <- colMeans(logR)  
10  
11 sig.hat <- cov(logR)
```

2 Question 2 (39 marks)

2.1 Question 2(a) (10 marks)

Using the provided $\hat{\Sigma}^{-1}$, $\hat{\Sigma}$, and $\hat{\mu}$ only:

Compute parameters:

$$\begin{aligned} c_{11} &:= \mathbf{1}^T \hat{\Sigma}^{-1} \mathbf{1} = 0.296 \\ c_{1\mu} &:= \mathbf{1}^T \hat{\Sigma}^{-1} \hat{\mu} = 0.01613 \\ c_{\mu\mu} &:= \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu} = 0.0011742 \\ \Delta &:= c_{11}c_{\mu\mu} - c_{1\mu}^2 = 0.00008739 \end{aligned}$$

Set the target expected return as $\mu_* = 0.07$, the variance-minimized portfolio is

$$\mathbf{w}_*(\mu_*) := \Delta^{-1} \left(c_{11} \hat{\Sigma}^{-1} \hat{\mu} - c_{1\mu} \hat{\Sigma}^{-1} \mathbf{1} \right) \mu_* + \Delta^{-1} \left(c_{\mu\mu} \hat{\Delta}^{-1} \mathbf{1} - c_{1\mu} \hat{\Sigma}^{-1} \hat{\mu} \right) = \begin{pmatrix} -0.8393 \\ 0.7007 \\ 0.5473 \\ 0.5914 \end{pmatrix}$$

The volatility is

$$\sigma_*(\mu_*) = \sqrt{\mathbf{w}_*^T \hat{\Sigma} \mathbf{w}_*} = \sqrt{\Delta^{-1} (c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu})} = 2.04765$$

Using the provided $\hat{\Sigma}^{-1} \mathbf{1}$ and $\hat{\Sigma}^{-1} \hat{\mu}$:

Compute parameters:

$$\begin{aligned} c_{11} &:= \mathbf{1}^T \hat{\Sigma}^{-1} \mathbf{1} = 0.32 \\ c_{1\mu} &:= \mathbf{1}^T \hat{\Sigma}^{-1} \hat{\mu} = 0.0172 \\ c_{\mu\mu} &:= \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu} = 0.001188 \\ \Delta &:= c_{11}c_{\mu\mu} - c_{1\mu}^2 = 0.00008432 \end{aligned}$$

Set the target expected return as $\mu_* = 0.07$, the variance-minimized portfolio is

$$\mathbf{w}_*(\mu_*) := \Delta^{-1} \left(c_{11} \hat{\Sigma}^{-1} \hat{\mu} - c_{1\mu} \hat{\Sigma}^{-1} \mathbf{1} \right) \mu_* + \Delta^{-1} \left(c_{\mu\mu} \hat{\Delta}^{-1} \mathbf{1} - c_{1\mu} \hat{\Sigma}^{-1} \hat{\mu} \right) = \begin{pmatrix} -0.8425 \\ 0.7059 \\ 0.5446 \\ 0.5920 \end{pmatrix}$$

The volatility is

$$\sigma_*(\mu_*) = \sqrt{\mathbf{w}_*^T \hat{\Sigma} \mathbf{w}_*} = \sqrt{\Delta^{-1} (c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu})} = 2.03154$$

2.2 Question 2(b) (10 marks)

Using the provided $\hat{\Sigma}^{-1}$, $\hat{\Sigma}$, and $\hat{\mu}$ only:

The efficient frontier is the upper part of the minimum-variance set, that is:

$$\begin{aligned}\sigma_*^2(\mu_*) &= \Delta^{-1} \left[c_{11} \left(\mu_* - \frac{c_{1\mu}}{c_{11}} \right)^2 + c_{\mu\mu} - \frac{c_{1\mu}^2}{c_{11}} \right] \\ &= 3387.259 (\mu_* - 0.054)^2 + 3.378, \quad \mu_* \geq \mu_{\minvar} = 0.0545 \\ &\stackrel{or}{=} 3387.259 \mu_*^2 - 369.165 \mu_* + 13.437, \quad \mu_* \geq \mu_{\minvar} = 0.0545\end{aligned}$$

Using the provided $\hat{\Sigma}^{-1} \mathbf{1}$ and $\hat{\Sigma}^{-1} \hat{\mu}$:

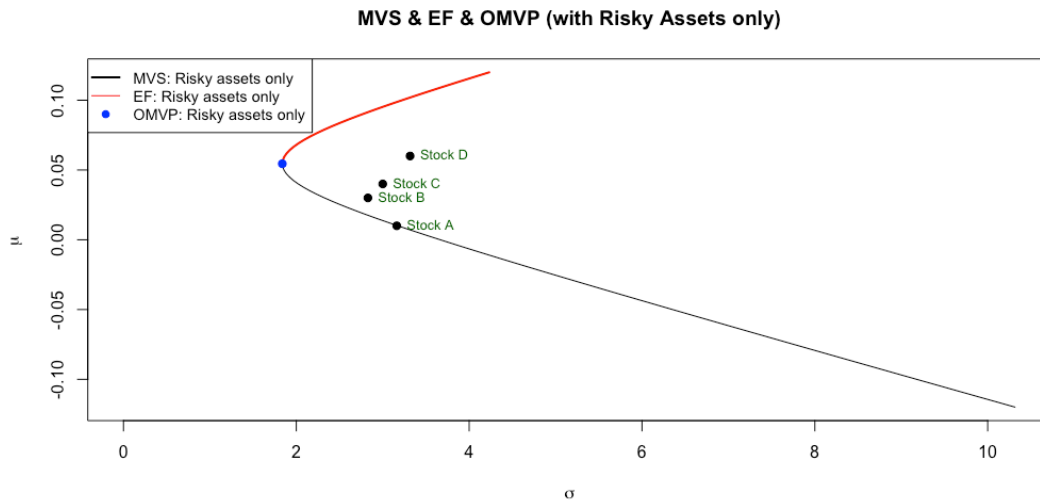
The efficient frontier is the upper part of the minimum-variance set, that is:

$$\begin{aligned}\sigma_*^2(\mu_*) &= \Delta^{-1} \left[c_{11} \left(\mu_* - \frac{c_{1\mu}}{c_{11}} \right)^2 + c_{\mu\mu} - \frac{c_{1\mu}^2}{c_{11}} \right] \\ &= 3795.066 (\mu_* - 0.054)^2 + 3.125, \quad \mu_* \geq \mu_{\minvar} = 0.0538 \\ &\stackrel{or}{=} 3795.066 \mu_*^2 - 407.97 \mu_* + 14.089, \quad \mu_* \geq \mu_{\minvar} = 0.0538\end{aligned}$$

Note that

- $\mu_A = 0.01, \sigma_A = \sqrt{10} \approx 3.162278$.
- $\mu_B = 0.03, \sigma_B = \sqrt{8} \approx 2.828427$.
- $\mu_C = 0.04, \sigma_C = \sqrt{9} = 3$.
- $\mu_D = 0.06, \sigma_D = \sqrt{11} \approx 3.316625$.

Hence, in the plot, $\mu_A < \mu_B < \mu_C < \mu_D$. However, $\sigma_B < \sigma_C < \sigma_A < \sigma_D$.



Remark (Question 2(b) (10 marks))

- (σ_a, μ_a) is on the MVS.
- no OMVP on the graph and no step for OMVP is fine, but you have to show me the equation correctly for the efficient frontier, e.g., $\mu_* \geq \mu_{\minvar} = 0.0545 \approx 0.0538$ or $\sigma_* \geq \sigma_{\minvar} = 1.838 \approx 1.768$.

2.3 Question 2(c) (4 marks)

Using the provided $\hat{\Sigma}^{-1}$, $\hat{\Sigma}$, and $\hat{\mu}$ only:

Note that:

$$w_{\text{minvar}} = \frac{\hat{\Sigma}^{-1} \mathbf{1}}{c_{11}} = \begin{pmatrix} -0.4054 \\ 0.5912 \\ 0.4020 \\ 0.4122 \end{pmatrix}$$

The expected return and volatility of the overall MVP can be found by

$$\mu_{\text{minvar}} = \frac{c_{1\mu}}{c_{11}} = 0.0545, \quad \text{and} \quad \sigma_{\text{minvar}} = \sqrt{\frac{1}{c_{11}}} = 1.838$$

Using the provided $\hat{\Sigma}^{-1} \mathbf{1}$ and $\hat{\Sigma}^{-1} \hat{\mu}$:

Note that:

$$w_{\text{minvar}} = \frac{\hat{\Sigma}^{-1} \mathbf{1}}{c_{11}} = \begin{pmatrix} -0.3438 \\ 0.5625 \\ 0.3750 \\ 0.4062 \end{pmatrix}$$

The expected return and volatility of the overall MVP can be found by

$$\mu_{\text{minvar}} = \frac{c_{1\mu}}{c_{11}} = 0.0538, \quad \text{and} \quad \sigma_{\text{minvar}} = \sqrt{\frac{1}{c_{11}}} = 1.768$$

Remark (Question 2(c) (4 marks))

- Without μ_{minvar} and σ_{minvar} is fine.

2.4 Question 2(d) (5 marks)

```

1 library(quadprog)
2 n = 4
3 muStar = 0.07
4 M = c(0.01,0.03,0.04,0.06)
5 C = matrix(c(10, 5, 6, 4,
6             5, 8, 2, -1,
7             6, 2, 9, 2,
8             4, -1, 2, 11),
9           byrow = TRUE, nrow = 4, ncol = 4)
10
11 V1 = matrix(1,n,1)
12 V0 = matrix(0,n,1)
13 Amat = cbind(M,V1,diag(n))
14 b0 = matrix(c(muStar,1,V0),n+2,1) ## n+2) without 1 is fine
15 p = solve.QP(C, V0, Amat, b0, 2)
16 W0 = p$solution
    
```

2.5 Question 2(e) (5 marks)

Using the provided $\hat{\Sigma}^{-1}$, $\hat{\Sigma}$, and $\hat{\mu}$ only:

The tangency portfolio is given by

$$w_M = \frac{\hat{\Sigma}^{-1}(\hat{\mu} - r\mathbf{1})}{c_{1\mu} - rc_{11}} = \begin{pmatrix} -1.5448 \\ 0.8786 \\ 0.7834 \\ 0.8828 \end{pmatrix},$$

$$\text{Expected return: } \mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}} = 0.0952,$$

$$\text{Volatility: } \sigma_M = \frac{|\gamma|}{c_{1\mu} - rc_{11}} = 2.9992, \quad (\sigma_M^2 \approx 9)$$

$$\text{where } \gamma = \sqrt{(\hat{\mu} - r\mathbf{1})^\top \hat{\Sigma}^{-1}(\hat{\mu} - r\mathbf{1})} = 0.0217.$$

Using the provided $\hat{\Sigma}^{-1}\mathbf{1}$ and $\hat{\Sigma}^{-1}\hat{\mu}$:

The tangency portfolio is given by

$$w_M = \frac{\hat{\Sigma}^{-1}(\hat{\mu} - r\mathbf{1})}{c_{1\mu} - rc_{11}} = \begin{pmatrix} -1.4079 \\ 0.8684 \\ 0.7368 \\ 0.8026 \end{pmatrix},$$

$$\text{Expected return: } \mu_M = \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}} = 0.0884,$$

$$\text{Volatility: } \sigma_M = \frac{|\gamma|}{c_{1\mu} - rc_{11}} = 2.8005, \quad (\sigma_M^2 \approx 7.84)$$

$$\text{where } \gamma = \sqrt{(\hat{\mu} - r\mathbf{1})^\top \hat{\Sigma}^{-1}(\hat{\mu} - r\mathbf{1})} = 0.0213.$$

2.6 Question 2(f) (5 marks)

1. **Zero Proportion:** If a security has a zero proportion in the tangency portfolio, it means no investor demands that security.
2. **No Demand:** A lack of demand for the security will lead to its price dropping.
3. **Price Drop:** When the price of the security drops, its expected return will increase.
4. **Expected Return Rises:** An increase in the expected return makes the security more attractive for investors.
5. **Included in Tangency Portfolio:** This increased attractiveness ensures that the security is included in the tangency portfolio.

3 Question 3 (17 marks)

3.1 Question 3(a) (4 marks)

Denote w_i as the market capitalization of asset i , $q_\alpha = \alpha$ -quantile of book-to-market ratios $\{z_1, z_2, \dots, z_n\}$ of the n -assets, R_{it} as the return of asset i . Then, we can calculate:

$$| R_{High,t} = \frac{\sum_{i=1}^n w_i \mathbb{1}_{\{z_i > q_{0.7}\}} R_{it}}{\sum_{i=1}^n w_i \mathbb{1}_{\{z_i > q_{0.7}\}}} \quad | R_{Low,t} = \frac{\sum_{i=1}^n w_i \mathbb{1}_{\{z_i \leq q_{0.3}\}} R_{it}}{\sum_{i=1}^n w_i \mathbb{1}_{\{z_i \leq q_{0.3}\}}}$$

With the above calculation, we can show the computation $HML_t = R_{High,t} - R_{Low,t}$.

3.2 Question 3(b) (4 marks)

$$R_{at} - r_t = 0.002 + 0.981(ReM) + 0.422(SMB) + 1.935(HML) + e_{at}$$

OR

$$R_{at} - r_t = 0.002 + 0.981(R_{Mt} - r_t) + 0.422(SMB) + 1.935(HML) + e_{at}$$

The factor loadings are $\hat{b} = \begin{pmatrix} \hat{b}_{A,1} \\ \hat{b}_{A,2} \\ \hat{b}_{A,3} \end{pmatrix} = \begin{pmatrix} 0.981 \\ 0.422 \\ 1.935 \end{pmatrix}$.

3.3 Question 3(c) (2 marks)

Note that the slope of Capital Market Line (CML) is called market price of risk, i.e.,

$$\text{Market Price of Risk: } \frac{\mu_M - r}{\sigma_M} = 0.0217 \quad (\text{or } 0.0209), \quad \text{by Question 2(e) and the given } r = 3\%.$$

Note that the slope of Security Market Line (SML) is called market risk premium, i.e.,

$$\text{Market Price of Risk: } \mu_M - r = 0.0652 \quad (\text{or } 0.0584), \quad \text{by Question 2(e) and the given } r = 3\%.$$

3.4 Question 3(d) (3 marks)

▪ Jensen index: $\hat{\alpha}_A$ $= 0.003 = 0.3\%$.	▪ Sharpe ratio: $\frac{\hat{\mu}_A - r}{\hat{\sigma}_A}$ $= \frac{0.01 - 0.03}{\sqrt{10}} = -0.00632$.	▪ Treynor ratio: $\frac{\hat{\mu}_A - r}{\hat{\beta}_A}$ $= \frac{0.01 - 0.03}{1.12} = -0.0179$.
---	--	--

3.5 Question 3(e) (4 marks)

Consider w , B and Ω_F as the portfolio weight, factor loadings, and the covariance of the factors. We have:

$$w := \begin{pmatrix} w_{rf} \\ w_A \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}, B := \begin{pmatrix} b_{rf,ReM} & b_{rf,SMB} & b_{rf,HML} \\ b_{A,ReM} & b_{A,SMB} & b_{A,HML} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0.981 & 0.422 & 1.935 \end{pmatrix}, \Omega_F = \begin{pmatrix} 0.16 & 0.036 & 0.17 \\ 0.036 & 0.012 & 0.045 \\ 0.17 & 0.045 & 0.22 \end{pmatrix}$$

Hence, the systematic risk is $\$2M \times w^\top B \Omega_F B^\top w = \$2M \times 62.22752\% \approx \$1.2446M$.

OR

Consider

$$w := 0.6, B := \hat{b}^\top = (0.981 \quad 0.422 \quad 1.935), \Omega_F = \begin{pmatrix} 0.16 & 0.036 & 0.17 \\ 0.036 & 0.012 & 0.045 \\ 0.17 & 0.045 & 0.22 \end{pmatrix}$$

Hence, the systematic risk is $\$2M \times w^\top B \Omega_F B^\top w = \$2M \times 62.22752\% \approx \$1.2446M$.

4 Question 4 (24 marks)

4.1 Question 4(a) (4 marks)

Given that

- $R \sim \mathcal{N}(\mu, \Sigma)$

We have:

- $\mathbb{E}(w^\top R) = w^\top \mathbb{E}(R) = w^\top \mu, \text{Var}(w^\top R) = w^\top \Sigma w.$
- i.e., $w^\top R \sim \mathcal{N}(w^\top \mu, w^\top \Sigma w).$

Hence:

- $\mathbb{P}(w^\top R \leq L) = \mathbb{P}\left(\frac{w^\top R - w^\top \mu}{\sqrt{w^\top \Sigma w}} \leq \frac{L - w^\top \mu}{\sqrt{w^\top \Sigma w}}\right) = \mathbb{P}\left(Z \leq \frac{L - w^\top \mu}{\sqrt{w^\top \Sigma w}}\right) = \Phi\left(\frac{L - w^\top \mu}{\sqrt{w^\top \Sigma w}}\right),$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution.

4.2 Question 4(b) (2 marks)

- $\Phi(\cdot)$ is monotonically increasing, i.e.,

$$\min_w \Phi\left(\frac{L - w^\top \mu}{\sqrt{w^\top \Sigma w}}\right) = \min_w \frac{L - w^\top \mu}{\sqrt{w^\top \Sigma w}}$$

- With the constraint $w^\top \mathbf{1} = 1$, we have:

$$\min_w \frac{L - w^\top \mu}{\sqrt{w^\top \Sigma w}} = \min_w \frac{L \times \mathbf{1} - w^\top \mu}{\sqrt{w^\top \Sigma w}} = \min_w \frac{L \times w^\top \mathbf{1} - w^\top \mu}{\sqrt{w^\top \Sigma w}} = \min_w \frac{w^\top (L\mathbf{1} - \mu)}{\sqrt{w^\top \Sigma w}}$$

4.3 Question 4(c) (6 marks)

Note that

$$\begin{aligned} \frac{\partial}{\partial w} \left(\frac{w^\top (L\mathbf{1} - \mu)}{\sqrt{w^\top \Sigma w}} \right) &= \frac{\sqrt{w^\top \Sigma w} \cdot (L\mathbf{1} - \mu) - w^\top (L\mathbf{1} - \mu) \cdot \frac{1}{2} \frac{1}{\sqrt{w^\top \Sigma w}} 2\Sigma w}{w^\top \Sigma w} \\ &= \frac{w^\top \Sigma w \cdot (L\mathbf{1} - \mu) - w^\top (L\mathbf{1} - \mu) \cdot \Sigma w}{(w^\top \Sigma w)^{3/2}} =: f'(w) \end{aligned}$$

Consider $f'(w_*) = 0$, we have

$$\begin{aligned} w_*^\top \Sigma w_* \cdot (L\mathbf{1} - \mu) - w_*^\top (L\mathbf{1} - \mu) \cdot \Sigma w_* &= 0 \\ w_*^\top \Sigma w_* \cdot (L\mathbf{1} - \mu) &= w_*^\top (L\mathbf{1} - \mu) \cdot \Sigma w_* \\ \underbrace{\frac{w_*^\top \Sigma w_*}{w_*^\top (L\mathbf{1} - \mu)} \cdot (L\mathbf{1} - \mu)}_{w_*^\top \Sigma w_* \in \mathbb{R}, \quad w_*^\top (L\mathbf{1} - \mu) \in \mathbb{R}} &= \Sigma w_* \\ \underbrace{\frac{w_*^\top \Sigma w_*}{w_*^\top (L\mathbf{1} - \mu)} \cdot \Sigma^{-1} (L\mathbf{1} - \mu)}_{(w_*^\top \Sigma w_*) / (w_*^\top (L\mathbf{1} - \mu)) \in \mathbb{R}} &= w_* \end{aligned}$$

Hence,

- the optimal w satisfies $w = c\Sigma^{-1}(L\mathbf{1} - \mu)$ for some constant c .

4.4 Question 4(d) (5 marks)

Consider

$$\mathcal{L}(w, \lambda) = \frac{w^\top (L1 - \mu)}{\sqrt{w^\top \Sigma w}} - \lambda(w^\top 1 - 1), \quad (\sqrt{w^\top \Sigma w} \neq 0)$$

$$\frac{\partial \mathcal{L}}{\partial w} = \frac{w^\top \Sigma w \cdot (L1 - \mu) - w^\top (L1 - \mu) \cdot \Sigma w}{(w^\top \Sigma w)^{3/2}} - \lambda(1), \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 1 - w^\top 1$$

Set them to zero and we will get:

$$\frac{w^\top \Sigma w \cdot (L1 - \mu) - w^\top (L1 - \mu) \cdot \Sigma w}{(w^\top \Sigma w)^{3/2}} = \lambda(1), \quad w^\top 1 = 1$$

i.e.,

$$\begin{aligned} w^\top \Sigma w \cdot (L1 - \mu) - w^\top (L1 - \mu) \cdot \Sigma w &= (w^\top \Sigma w)^{3/2} \lambda(1) \\ \textcolor{red}{w}^\top w^\top \Sigma w \cdot (L1 - \mu) - \textcolor{red}{w}^\top w^\top (L1 - \mu) \cdot \Sigma w &= \textcolor{red}{w}^\top (w^\top \Sigma w)^{3/2} \lambda(1) \\ w^\top \Sigma w \cdot (L\textcolor{red}{w}^\top 1 - \textcolor{red}{w}^\top \mu) - w^\top (L1 - \mu) \cdot \textcolor{red}{w}^\top \Sigma w &= (w^\top \Sigma w)^{3/2} \lambda(\textcolor{red}{w}^\top 1) \\ w^\top \Sigma w \cdot (L - \textcolor{red}{w}^\top \mu) - w^\top (L1 - \mu) \cdot \textcolor{red}{w}^\top \Sigma w &= (w^\top \Sigma w)^{3/2} \lambda \\ (L - w^\top \mu) - w^\top (L1 - \mu) &= \lambda \sqrt{w^\top \Sigma w}, \quad (w^\top \Sigma w \in \mathbb{R}) \\ (L - w^\top \mu) - (Lw^\top 1 - w^\top \mu) &= \lambda \sqrt{w^\top \Sigma w} = (L - w^\top \mu) - (L - w^\top \mu) = 0 \\ \lambda &= 0 \end{aligned}$$

Using Question 4(c), we know that $w^\top \Sigma w \cdot (L1 - \mu) - w^\top (L1 - \mu) \cdot \Sigma w = 0$ implies:

$$c = \frac{w^\top \Sigma w}{w^\top (L1 - \mu)} \stackrel{w^\top 1=1}{=} \frac{w^\top \Sigma w}{L - w^\top \mu}, \quad w = c \Sigma^{-1} (L1 - \mu).$$

Note that

$$\begin{cases} w^\top \mu &= (c \Sigma^{-1} (L1 - \mu))^\top = c(L1 - \mu)^\top \Sigma^{-1} \mu := c(Lc_{1\mu} - c_{\mu\mu}) = c(c_{1\mu}L - c_{\mu\mu}) \\ w^\top \Sigma w &= (c \Sigma^{-1} (L1 - \mu))^\top \Sigma (c \Sigma^{-1} (L1 - \mu)) = c^2 (L1 - \mu)^\top \Sigma^{-1} (L1 - \mu) = c^2 (c_{11}L^2 - 2c_{1\mu}L + c_{\mu\mu}) \end{cases}$$

where

$$\begin{aligned} (L1 - \mu)^\top \Sigma^{-1} (L1 - \mu) &= (L1^\top - \mu^\top) \Sigma^{-1} (L1 - \mu) \\ &= (L1^\top \Sigma^{-1} - \mu^\top \Sigma^{-1}) (L1 - \mu) \\ &= (1^\top \Sigma^{-1} 1) L^2 - (1^\top \Sigma^{-1} \mu + \mu^\top \Sigma^{-1} 1) L + \mu^\top \Sigma^{-1} \mu \\ &:= c_{11}L^2 - 2c_{\mu 1}L + c_{\mu\mu} \stackrel{\text{or}}{=} c_{11}L^2 - 2c_{1\mu}L + c_{\mu\mu} \end{aligned}$$

Hence, we have: (Remark: This is meaningless for you to consider $L = 0$ for the portfolio return.)

$$c = \frac{c^2 (c_{11}L^2 - 2c_{1\mu}L + c_{\mu\mu})}{L - c(c_{1\mu}L - c_{\mu\mu})} \stackrel{c \neq 0}{\Rightarrow} L - c(c_{1\mu}L - c_{\mu\mu}) = c(c_{11}L^2 - 2c_{1\mu}L + c_{\mu\mu}) \stackrel{L \neq 0}{\Rightarrow} 1 = c(c_{11}L - c_{1\mu})$$

i.e.,

$$c = \frac{1}{c_{11}L - c_{1\mu}}, \quad w = \frac{1}{c_{11}L - c_{1\mu}} \Sigma^{-1} (L1 - \mu) = \frac{\Sigma^{-1} (L1 - \mu)}{c_{11}L - c_{1\mu}}, \quad (c_{11}L - c_{1\mu} \neq 0).$$

OR

Considering $w^\top 1 = 1^\top w = 1$, we have $1 = 1^\top w = 1^\top (c \Sigma^{-1} (L1 - \mu)) = c(c_{11}L - c_{1\mu})$. Hence, we have:

$$c = \frac{1}{c_{11}L - c_{1\mu}}, \quad w = \frac{1}{c_{11}L - c_{1\mu}} \Sigma^{-1} (L1 - \mu) = \frac{\Sigma^{-1} (L1 - \mu)}{c_{11}L - c_{1\mu}}, \quad (c_{11}L - c_{1\mu} \neq 0).$$

4.5 Question 4(e) (5 marks)

Note that $w^\top \Sigma w \in \mathbb{R}$, we get:

$$A := \frac{L - w^\top \mu}{\sqrt{w^\top \Sigma w}} = \frac{L - w^\top \mu}{w^\top \Sigma w} \sqrt{w^\top \Sigma w} = \underbrace{\frac{1}{c}}_{\text{Question 4(d)}} \sqrt{w^\top \Sigma w}$$

Using $w = c\Sigma^{-1}(L1 - \mu)$, we have:

$$\begin{cases} w^\top \Sigma w &= (c\Sigma^{-1}(L1 - \mu))^\top \Sigma (c\Sigma^{-1}(L1 - \mu)) = c^2(L1 - \mu)^\top \Sigma^{-1}(L1 - \mu) \\ \sqrt{w^\top \Sigma w} &= (c\Sigma^{-1}(L1 - \mu))^\top \Sigma (c\Sigma^{-1}(L1 - \mu)) = |c| \sqrt{(L1 - \mu)^\top \Sigma^{-1}(L1 - \mu)} \end{cases}$$

Hence, we get:

$$A = \frac{1}{c} |c| \sqrt{(L1 - \mu)^\top \Sigma^{-1}(L1 - \mu)} \stackrel{\text{Question 4(d)}}{=} \frac{|c|}{c} \sqrt{c_{11}L^2 - 2c_{1\mu}L + c_{\mu\mu}} \stackrel{\text{or}}{=} \frac{|c|}{c} \sqrt{c_{11}L^2 - 2c_{\mu 1}L + c_{\mu\mu}}$$

Note that $c = \frac{1}{c_{11}L - c_{1\mu}} \in \mathbb{R} \setminus \{0\}$, i.e., c can be positive or negative.

Overall, we have

▪

$$a = \frac{|c|}{c} = \begin{cases} 1, & \text{if } c > 0 \text{ or } L > \frac{c_{1\mu}}{c_{11}}; \\ -1, & \text{if } c < 0 \text{ or } L < \frac{c_{1\mu}}{c_{11}}; \\ \text{undefined}, & \text{if } c \text{ is undefined or } L = \frac{c_{1\mu}}{c_{11}}. \end{cases} \stackrel{\text{or}}{=} \begin{cases} 1, & \text{if } c > 0 \text{ or } L > \frac{c_{1\mu}}{c_{11}}; \\ -1, & \text{if } c < 0 \text{ or } L < \frac{c_{1\mu}}{c_{11}}; \\ \text{DNE}, & \text{if } c \text{ DNE or } L = \frac{c_{1\mu}}{c_{11}}. \end{cases}$$

▪ **Remark:** $a \neq \text{sgn}(c_{11}L - c_{1\mu})$ because when $c_{11}L - c_{1\mu} = 0$, c is undefined instead of 0.

▪ $b_1 = \mathbf{1}^\top \Sigma^{-1} \mathbf{1} = c_{11}.$

▪ $b_2 = \mathbf{1}^\top \Sigma^{-1} \mu + \mu^\top \Sigma^{-1} \mathbf{1} = c_{1\mu} + c_{\mu 1} = 2c_{1\mu} = 2c_{\mu 1}.$

▪ $b_3 = \mu^\top \Sigma^{-1} \mu = c_{\mu\mu}.$

▪ such that

$$A = \begin{cases} \sqrt{b_1 L^2 - b_2 L + b_3}, & \text{if } c > 0 \text{ or } a = 1; \\ -\sqrt{b_1 L^2 - b_2 L + b_3}, & \text{if } c < 0 \text{ or } a = -1; \\ \text{undefined}, & \text{if } c \text{ is undefined or } a \text{ is undefined.} \end{cases} \stackrel{\text{or}}{=} \begin{cases} \sqrt{b_1 L^2 - b_2 L + b_3}, & \text{if } c > 0 \text{ or } a = 1; \\ -\sqrt{b_1 L^2 - b_2 L + b_3}, & \text{if } c < 0 \text{ or } a = -1; \\ \text{DNE}, & \text{if } c \text{ DNE or } a \text{ DNE.} \end{cases}$$

Remark: DNE = Does not exist.

4.6 Question 4(f) (2 marks)

Note that $b_1, b_2, b_3 > 0$, and $\Phi(\cdot)$ is monotonic increasing, such that $\min_L \Phi(A) = \min_L(A)$.

$$f(L) := \sqrt{b_1 L^2 - b_2 L + b_3} > 0, \quad \frac{\partial f(L)}{\partial L} = 0 \iff \frac{\partial f^2(L)}{\partial L} = 0 = 2b_1 L - b_2$$

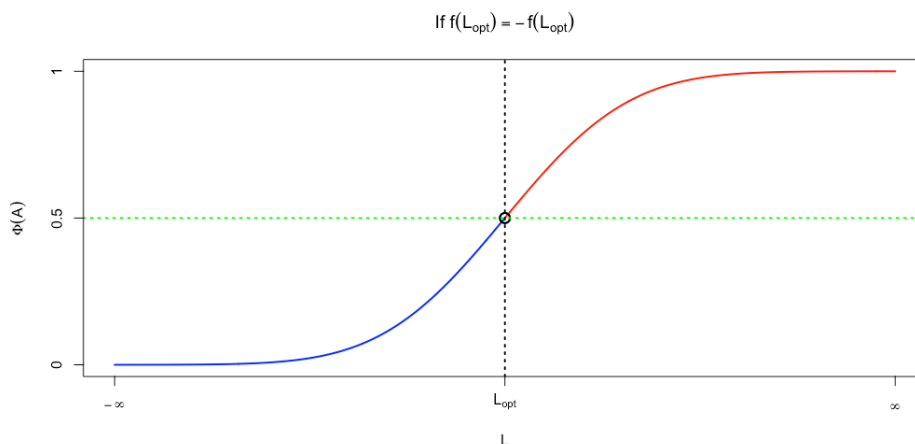
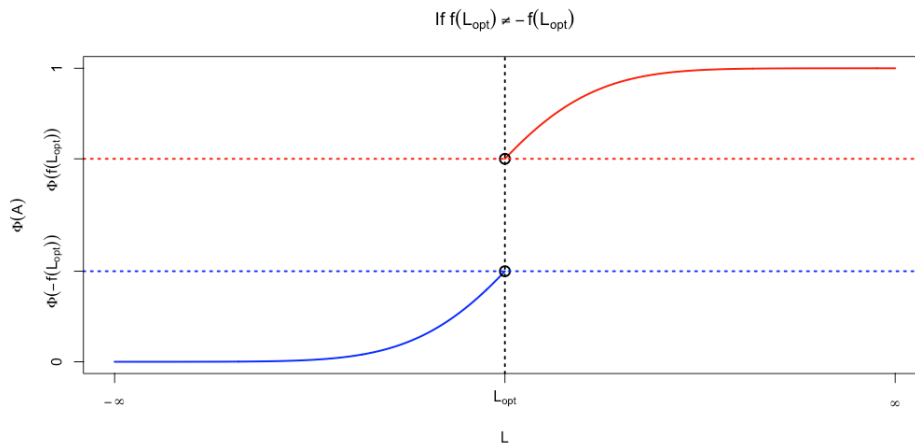
With $b_1 > 0$, $L_{\text{opt}} := \frac{b_2}{2b_1} = \frac{c_{1\mu}}{c_{11}} \stackrel{\text{or}}{=} \frac{c_{\mu 1}}{c_{11}}$. Also, $L_{\text{opt}} =: L_{\min}$ for $a = 1$ and $L_{\text{opt}} =: L_{\max}$ for $a = -1$, i.e.,

$$A \text{ is } \begin{cases} \text{decreasing,} & \text{if } L < L_{\text{opt}} \text{ and } a = 1; \\ \text{increasing,} & \text{if } L < L_{\text{opt}} \text{ and } a = -1; \\ \text{undefined,} & \text{if } a \text{ DNE.} \end{cases} \text{ and } A \text{ is } \begin{cases} \text{increasing,} & \text{if } L > L_{\text{opt}} \text{ and } a = 1; \\ \text{decreasing,} & \text{if } L > L_{\text{opt}} \text{ and } a = -1; \\ \text{undefined,} & \text{if } a \text{ DNE.} \end{cases}$$

When $L < L_{\text{opt}}$, by Question 4(e), $a = -1$, A is increasing. When $L = L_{\text{opt}}$, by Question 4(e), a DNE, A DNE. When $L > L_{\text{opt}}$, by Question 4(e), $a = 1$, A is increasing. Again, by Question 4(e), we have

$$A = \begin{cases} \sqrt{b_1 L^2 - b_2 L + b_3}, & \text{if } a = 1; \\ -\sqrt{b_1 L^2 - b_2 L + b_3}, & \text{if } a = -1; \\ \text{DNE,} & \text{if } a \text{ DNE.} \end{cases} = \begin{cases} f(L), & \text{if } a = 1; \\ -f(L), & \text{if } a = -1; \\ \text{DNE,} & \text{if } a \text{ DNE.} \end{cases} \text{ with } \Phi(A) = \begin{cases} \Phi(f(L)), & \text{if } a = 1; \\ \Phi(-f(L)), & \text{if } a = -1; \\ \text{DNE,} & \text{if } a \text{ DNE.} \end{cases}$$

So, the graph w.r.t. L starts from $(-\infty, L_{\text{opt}})$ in $\Phi(-f(L))$, jumps at L_{opt} , and continues to (L_{opt}, ∞) in $\Phi(f(L))$.



No minimum of A can be identified as well as $\Phi(A)$ by a finite L . Hence, no, we cannot. (Remark: You will be fine if you can draw the first graph.)

5 Question 5 (6 marks)

5.1 Question 5(a) (2 marks)

$\det(\Sigma) = 0$ implies that the rows (or columns) of Σ are linearly dependent.

For a covariance matrix Σ , let the i -th row of Σ be:

$$(\text{Cov}(R_i, R_1), \text{Cov}(R_i, R_2), \dots, \text{Cov}(R_i, R_p)).$$

Since $\det(\Sigma) = 0$, there exist constants $c_{j_1}, \dots, c_{j_k}$ such that:

$$\text{Cov}(R_i, R_j) = \sum_{m=1}^k c_{j_m} \text{Cov}(R_{j_m}, R_j), \quad \forall j \neq i.$$

With $\tilde{R}_i = \sum_{m=1}^k c_{j_m} R_{j_m}$, we have

$$\text{Cov}(\tilde{R}_i, R_j) = \text{Cov}\left(\sum_{m=1}^k c_{j_m} R_{j_m}, R_j\right) = \sum_{m=1}^k c_{j_m} \text{Cov}(R_{j_m}, R_j).$$

By construction:

$$\text{Cov}(\tilde{R}_i, R_j) = \text{Cov}(R_i, R_j),$$

which completes the proof.

5.2 Question 5(b) (2 marks)

The singularity of Σ ensures that R_i is a linear combination of $\{R_{j_1}, \dots, R_{j_k}\}$, i.e., $R_i = \sum_{m=1}^k c_{j_m} R_{j_m}$.

Note that

- $\mathbb{E}(\tilde{R}_i - R_i) = 0 = \mathbb{E}(\tilde{R}_i - R_i)^2$;
- $\text{Var}(\tilde{R}_i - R_i) = \text{Var}(\tilde{R}_i) + \text{Var}(R_i) - 2\text{Cov}(\tilde{R}_i, R_i) = \text{Var}(\tilde{R}_i) + \text{Var}(R_i) - 2\text{Var}(\tilde{R}_i) = 0$,

which is making $\mathbb{E}(\tilde{R}_i - R_i)^2 = 0$ because the projection $\tilde{R}_i$ perfectly reconstructs R_i . The statement is equivalent.

5.3 Question 5(c) (2 marks)

If $\mathbb{E}(\tilde{R}_i) > \mathbb{E}(R_i)$, in theory, the construction of the optimal portfolio will likely overestimate the attractiveness of R_i , and the price of the overvalued asset R_i will adjust to its true value with certainty.

- Short sell the overpriced asset R_i to get "free lunch", i.e., arbitrage opportunities.