2026-2027 RMSC4003 Midterm Solution

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Question 1

(a)

```
length(ReP)
```

(b)

- **SMB: Small Minus Big.** Market capitalization (share price × shares outstanding).
- **HML: High Minus Low.** Book-to-market equity ratio (book equity / market equity).
- **CMA: Conservative Minus Aggressive.** Investment, measured by the proportional growth in total assets.

(c)

```
q <- quantile(Z, c(0.3, 0.7))
robust <- Z > q[2]
weak <- Z <= q[1]

Rrobust <- Re[, robust, drop = FALSE] %*%
           (W[robust] / sum(W[robust]))
Rweak <- Re[, weak, drop = FALSE] %*%
         (W[weak] / sum(W[weak]))
RMW <- as.vector(Rrobust - Rweak)
```

(d)

```
fit <- lm(I(ReP - r) ~ I(ReM - r) + SMB + HML + RMW + CMA)
```

Question 2

We first compute the following:

$$c_{11} = \mathbf{1}^T \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1} \approx 19.0909, \qquad c_{1\mu} = \mathbf{1}^T \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\mu}} \approx 0.536364,$$
$$c_{\mu\mu} = \hat{\boldsymbol{\mu}}^T \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\mu}} \approx 0.0165455, \quad \Delta = c_{11} c_{\mu\mu} - c_{1\mu}^2 \approx 0.0281818.$$

(a) For $\mu_* = 0.03$, the minimum-variance portfolio of risky assets is

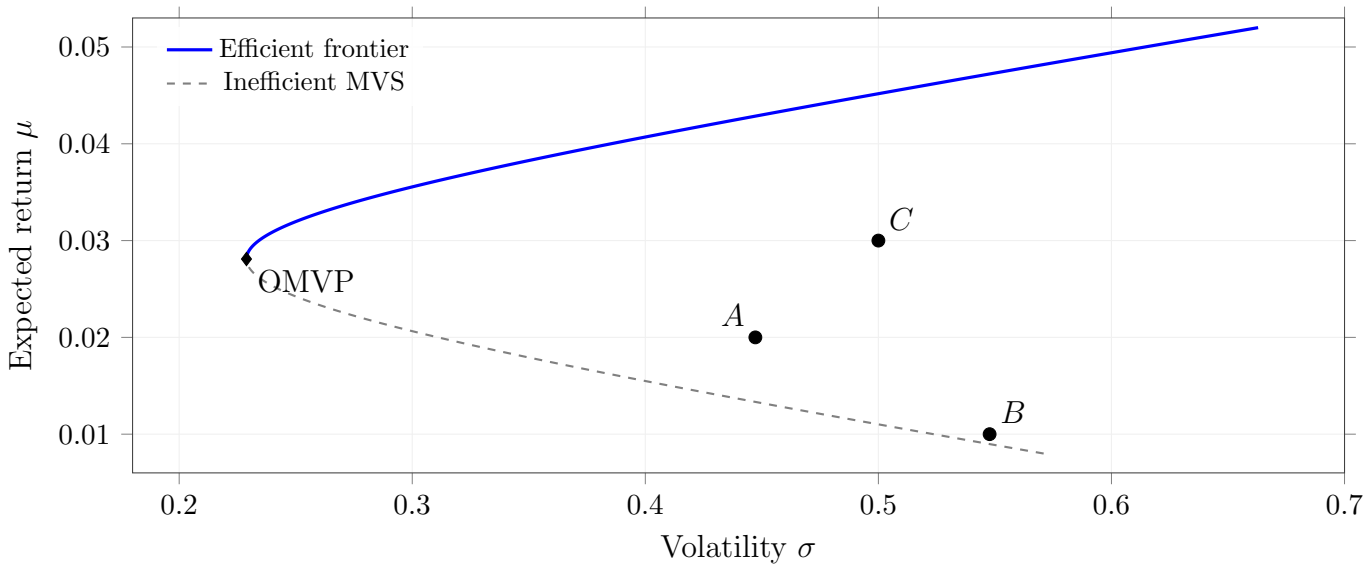
$$\begin{aligned}\boldsymbol{w}_*(\mu_*) &= \Delta^{-1}\big(c_{11}\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} - c_{1\mu}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\big)\mu_* \\ &\quad + \Delta^{-1}\big(c_{\mu\mu}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1} - c_{1\mu}\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}\big), \\ \boldsymbol{w}_*(0.03) &\approx \begin{pmatrix} 0.71 \\ -0.35 \\ 0.65 \end{pmatrix}, \\ \sigma_*^2(0.03) &= \boldsymbol{w}_*^T\hat{\boldsymbol{\Sigma}}\boldsymbol{w}_* \approx 0.055.\end{aligned}$$

(b) The minimum-variance set (MVS) satisfies

$$\sigma_*^2(\mu_*) = \Delta^{-1}\big(c_{11}\mu_*^2 - 2c_{1\mu}\mu_* + c_{\mu\mu}\big).$$

The upper branch, $\mu_* \geq \mu_{\text{minvar}} = c_{1\mu}/c_{11} \approx 0.028$, is the efficient frontier. The positions of Stocks A, B and C are

$$A: \ (\sqrt{0.20}, 0.02) \approx (0.45, 0.020), \quad B: \ (\sqrt{0.30}, 0.01) \approx (0.55, 0.010), \quad C: \ (0.50, 0.030).$$



(c) The overall minimum-variance risky portfolio is

$$\begin{aligned}\boldsymbol{w}_{\text{minvar}} &= \frac{\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}{c_{11}} \approx \begin{pmatrix} 0.67 \\ -0.24 \\ 0.57 \end{pmatrix}, \\ \mu_{\text{minvar}} &= \frac{c_{1\mu}}{c_{11}} \approx 0.028, \qquad \sigma^2_{\text{min}} = \frac{1}{c_{11}} \approx 0.052.\end{aligned}$$

(d) We minimize $\boldsymbol{w}^T\hat{\boldsymbol{\Sigma}}\boldsymbol{w}$ subject to $\boldsymbol{w}^T\hat{\boldsymbol{\mu}} = 0.025$, $\boldsymbol{w}^T\mathbf{1} = 1$, and $\boldsymbol{w} \geq \mathbf{0}$. The R commands are

```
library(quadprog)
n <- 3
M <- c(0.02, 0.01, 0.03)
C <- matrix(c(0.20, 0.10, -0.10,
              0.10, 0.30,  0.10,
             -0.10, 0.10,  0.25), n, n, byrow=TRUE)
V1 <- matrix(1, n, 1)
V0 <- matrix(0, n, 1)
Amat <- cbind(M, V1, diag(n))
b0 <- matrix(c(0.025, 1, V0), n+2, 1)
p <- solve.QP(C, V0, Amat, b0, 2)
W0 <- p$solution
W0
```

The commands give

$$\boldsymbol{w}_* \approx (0.50, 0, 0.50)^T, \qquad \sigma_*^2 \approx 0.063$$

(e) With $r = 0.008$, we obtain

$$\begin{aligned}
\gamma^2 &= c_{\mu\mu} - 2rc_{1\mu} + r^2 c_{11} \approx 0.00918545, \\
\boldsymbol{w}_M &= \frac{\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r\mathbf{1})}{c_{1\mu} - rc_{11}} \approx \begin{pmatrix} 0.75 \\ -0.47 \\ 0.72 \end{pmatrix}, \\
\mu_M &= \frac{c_{\mu\mu} - rc_{1\mu}}{c_{1\mu} - rc_{11}} \approx 0.032, \\
\sigma_M &= \frac{|\gamma|}{c_{1\mu} - rc_{11}} \approx 0.25
\end{aligned}$$

(f) 1) *Efficient frontier / CML*; 2) *linear*; 3) *risk-free asset*; 4) *tangency portfolio.*

(g) Let α be the weight invested in the risk-free asset, and $1 - \alpha$ the weight invested in the tangency portfolio M.

For $\mu_* = 0.06$, we have

$$\begin{aligned}
\mu_* &= \alpha r + (1 - \alpha)\mu_M, \\
\alpha &= \frac{\mu_M - \mu_*}{\mu_M - r} \approx -1.2,
\end{aligned}$$

$$\widetilde{\boldsymbol{w}}_* = \begin{pmatrix} \alpha \\ (1-\alpha)\boldsymbol{w}_M \end{pmatrix} \approx \begin{pmatrix} -1.2 \\ 1.6 \\ -1.0 \\ 1.6 \end{pmatrix}$$

(h) For Stock A, the CAPM beta is

$$\beta_a = \frac{\mathrm{Cov}(R_a, R_M)}{\mathrm{Var}(R_M)} = \frac{(\hat{\boldsymbol{\Sigma}}\boldsymbol{w}_M)_1}{\boldsymbol{w}_M^T \hat{\boldsymbol{\Sigma}} \boldsymbol{w}_M} \approx \frac{0.03128}{0.06241} \approx 0.50.$$

(i) For Stock A, $\mu_a = 0.02$ and $\sigma_a = \sqrt{0.20}$, so

$$\text{Sharpe ratio} = \frac{\mu_a - r}{\sigma_a} = \frac{0.02 - 0.008}{\sqrt{0.20}} \approx 0.027,$$
$$\text{Treynor ratio} = \frac{\mu_a - r}{\beta_a} \approx \frac{0.02 - 0.008}{0.50119} \approx 0.024.$$

(j) Under the CAPM equilibrium assumptions, all investors hold the same tangency portfolio $\boldsymbol{w}_M$ as their risky fund. Aggregate demand must equal the market supply, so the fraction invested in Stock i is

$$w_{M,i} = w_i^{(m)} = \frac{p_i n_i}{\sum_{j=1}^{n} p_j n_j}.$$

Consequently, the tangency portfolio is the market portfolio in equilibrium. However, the computed portfolio has $w_{M,2} \approx -0.47 < 0$, whereas $p_2 n_2 > 0$ implies $w_2^{(m)} > 0$. Thus, the assumption in (h) is not consistent with the market-capitalization-weighted portfolio.

(k) For $P_0 = 1000(2.2) = 2200$, $M_0 = 120$ and $\beta_p = \beta_a$,

$$N = \frac{\beta_p P_0}{M_0} \approx \frac{0.50(2200)}{120} = 9.2.$$

Short approximately 9.2 market-portfolio futures to hedge the systematic risk.

Question 3

(a) As $w_1 + w_2 + w_3 = 1 \implies w_2 + w_3 = 1 - w_1$, we can substitute the objective variance function $\sigma^2(\boldsymbol{w}) = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} = w_1^2 + (w_2 + w_3)^2$ and rewrite it to

$$\sigma^2(w_1) = w_1^2 + (1 - w_1)^2$$

Differentiating with respect to w_1 to find the global minimum

$$\frac{d}{dw_1}\left(w_1^2 + (1 - w_1)^2\right) = 2w_1 - 2(1 - w_1) = 4w_1 - 2 = 0 \implies w_1^* = \frac{1}{2}$$

Hence, we get $w_2^* = -1$ and $w_3^* = \frac{3}{2}$. The optimal portfolio weight vector is

$$\boldsymbol{w}^* = \begin{pmatrix} w_1^* \\ w_2^* \\ w_3^* \end{pmatrix} = \begin{pmatrix} 0.5 \\ -1 \\ 1.5 \end{pmatrix}$$

The variance of this portfolio is

$$\sigma^2 = (w_1^*)^2 + (w_2^* + w_3^*)^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

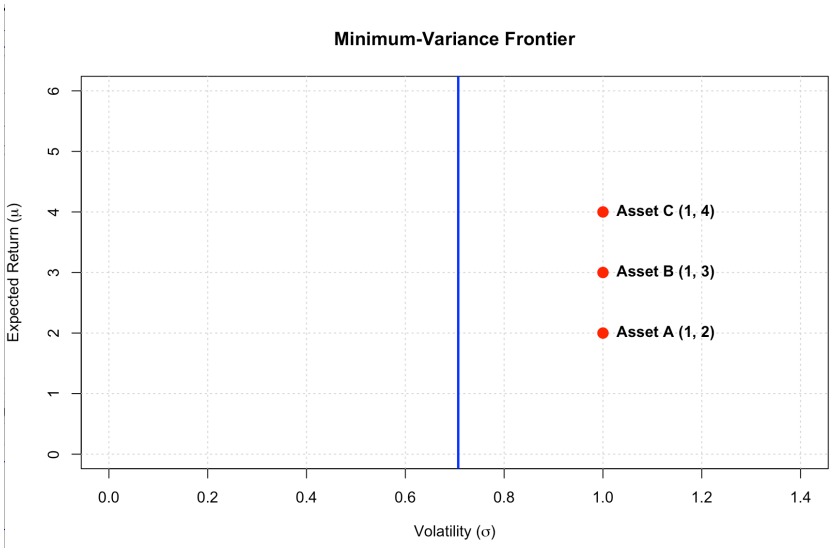
Hence, the volatility is $\sigma = \frac{\sqrt{2}}{2} \approx 0.7071$.

(b) In (a), we simplify the objective function simplifies to $\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w} = w_1^2 + (1 - w_1)^2$. This objective function depends **solely** on w_1 and is strictly convex. Its unconstrained minimum uniquely occurs at $w_1^* = \frac{1}{2}$, which implies $w_2^* + w_3^* = \frac{1}{2}$. The target return constraint can be simplified to

$$2w_1^* + 3w_2^* + 4w_3^* = 2\left(\frac{1}{2}\right) + 3(w_2^* + w_3^*) + w_3^* = \frac{5}{2} + w_3^* = \mu^*$$

which is free of w_1. Since the target return constraint can always be satisfied for any μ^* by setting $w_3 = \mu^* - \frac{5}{2}$ (without altering $w_1 = \frac{1}{2}$), the minimal variance is identically $\frac{1}{2}$ for all μ^*.

(c)



(d) Pascal's claim does not agree with the answer in a). In part a), the optimal weight is $w_2^* = -1 \neq 0$. Because Stocks B and C are perfectly correlated ($\rho_{BC} = 1$), an investor should short Stock B ($w_2^* < 0$) and go long Stock C ($w_3^* > 0$) to exploit the return spread while perfectly hedging the covariance risk, rather than abandoning Stock B.

(e) Jerry's claim does not agree with the answer in a). The optimal volatility found in part a) is $\sigma = \frac{\sqrt{2}}{2} > 0$, not 0. Under the budget constraint $\boldsymbol{w}^T\mathbf{1} = 1$, the portfolio variance is bounded below by $\sigma^2 = w_1^2 + (1 - w_1)^2 \geq \frac{1}{2}$, so the variance arising from Stock A and the aggregate $(w_2 + w_3)$ cannot be simultaneously eliminated.

(f) Parco's Claim is correct. The objective variance $\sigma^2 = w_1^2 + (1 - w_1)^2$ depends solely on w_1 and achieves its minimum at $w_1^* = \frac{1}{2}$, regardless of the asset mean vector $\boldsymbol{\mu}$. As long as $\mu_C \neq 3$, the target return constraint can always be absorbed by adjusting w_2^* and w_3^* while maintaining $w_1^* = \frac{1}{2}$, leaving the minimal volatility unchanged at $\frac{\sqrt{2}}{2}$.

NB. Stating that in (b) we have proved that the variance is free of μ is also acceptable.

(g) Let $\tilde{\boldsymbol{w}} = (\boldsymbol{w}^T, w_r)^T = (w_1, w_2, w_3, w_r)^T \in \mathbb{R}^4$ be the allocation vector. Because the risk-free asset has zero variance and zero covariance with all risky assets, the portfolio variance depends solely on the

risky assets

$$\sigma^2(\tilde{\boldsymbol{w}}) = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} = w_1^2 + (w_2 + w_3)^2$$

The optimization problem is

$$\begin{aligned} \arg\min_{\tilde{\boldsymbol{w}}} \quad & w_1^2 + (w_2 + w_3)^2 \\ \text{subject to} \quad & w_1 + w_2 + w_3 + w_r = 1, && \text{(Budget Constraint)} \\ & 2w_1 + 3w_2 + 4w_3 + r w_r = \mu^*, \quad \text{with } r = 0.5. && \text{(Target Return Constraint)} \end{aligned}$$

Following the Lagrangian multiplier approach from the lecture notes

$$L(\tilde{\boldsymbol{w}}, \lambda_\mu, \lambda_1) = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} - 2\lambda_\mu \left(\boldsymbol{w}^T \boldsymbol{\mu} + r w_r - \mu^* \right) - 2\lambda_1 \left(\boldsymbol{w}^T \mathbf{1} + w_r - 1 \right).$$

Setting the first-order derivatives to zero:

$$\frac{\partial L}{\partial w_r} = 2(-\lambda_\mu r - \lambda_1) = 0 \implies \lambda_1 = -r\lambda_\mu = -0.5\lambda_\mu, \tag{1}$$

$$\frac{\partial L}{\partial \boldsymbol{w}} = 2(\boldsymbol{\Sigma}\boldsymbol{w} - \lambda_\mu \boldsymbol{\mu} - \lambda_1 \mathbf{1}) = \mathbf{0} \implies \boldsymbol{\Sigma}\boldsymbol{w} = \lambda_\mu(\boldsymbol{\mu} - r\mathbf{1}). \tag{2}$$

Expanding Equation (2) using $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu} - r\mathbf{1} = (2 - 0.5,\, 3 - 0.5,\, 4 - 0.5)^T = (1.5,\, 2.5,\, 3.5)^T$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \lambda_\mu \begin{pmatrix} 1.5 \\ 2.5 \\ 3.5 \end{pmatrix}.$$

This yields the scalar system:

$$w_1 = 1.5\lambda_\mu, \tag{3}$$

$$w_2 + w_3 = 2.5\lambda_\mu, \tag{4}$$

$$w_2 + w_3 = 3.5\lambda_\mu. \tag{5}$$

Equating (4) and (5)

$$2.5\lambda_\mu = 3.5\lambda_\mu \implies \lambda_\mu = 0$$

Substituting $\lambda_\mu = 0$ back into (1), (3), and (4)

$$\lambda_1 = 0, \quad w_1^* = 0, \quad \text{and} \quad w_2^* + w_3^* = 0 \implies w_2^* = -w_3^*$$

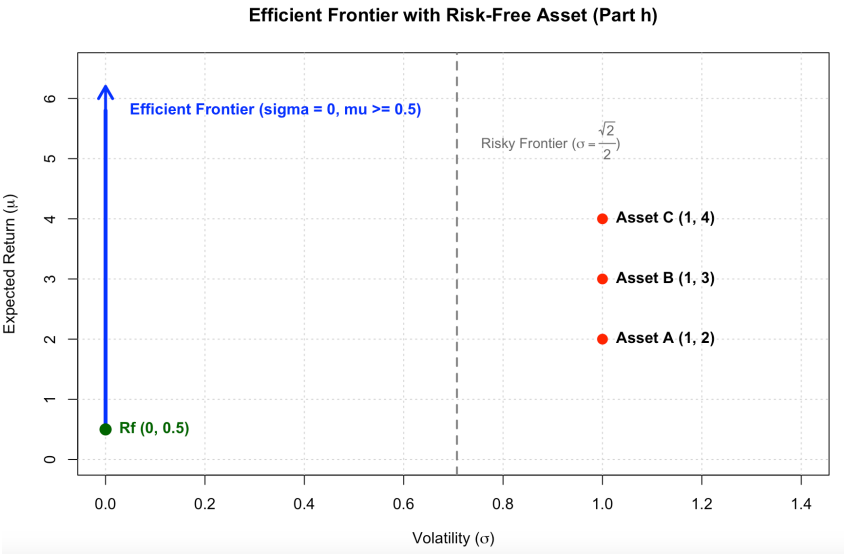
Now substitute these results into the two linear constraints, and we have $w_r^* = 1$, $w_3^* = \mu^* - 0.5$, and

$w_2^* = 0.5 - \mu^*$. The optimal portfolio weights are

$$\boldsymbol{w}^* = \begin{pmatrix} w_1^* \\ w_2^* \\ w_3^* \\ w_r^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0.5 - \mu^* \\ \mu^* - 0.5 \\ 1 \end{pmatrix}$$

The resulting minimal portfolio variance is $\sigma^2 = (w_1^*)^2 + (w_2^* + w_3^*)^2 = 0^2 + 0^2 = 0 \implies \sigma = 0$

(h)



Question 4

Yes, CAPM beta can be negative; **no**, it has no finite upper bound when borrowing and short-selling are unrestricted.

For a portfolio with weight w in the market portfolio and $1 - w$ in the risk-free asset,

$$R_p = wR_M + (1 - w)r.$$

It follows that

$$\begin{aligned} \beta_p &= \frac{\mathrm{Cov}(R_p, R_M)}{\mathrm{Var}(R_M)} = \frac{\mathrm{Cov}(wR_M + (1-w)r, R_M)}{\mathrm{Var}(R_M)} \\ &= \frac{w\mathrm{Var}(R_M)}{\mathrm{Var}(R_M)} = w. \end{aligned}$$

Therefore, $w < 0$ gives a negative beta, while $w \to \infty$ gives $\beta_p \to \infty$.