

RMSC 4003 Statistical Modelling in Financial Markets

Midterm

9:30-12:15. Tuesday, 25 Oct 2022

Name: _____

Major: _____

Part I: Applications: (70 marks):

Suppose that the risk-free rate is $r = 0.02$. We are interested in constructing an optimal portfolio using three stocks in the market, Stocks A, B and C.

- 1) (10 marks) Let the **log-returns** of the three stocks be stored in the vectors $\mathbf{L}_A$, $\mathbf{L}_B$, and $\mathbf{L}_C$, respectively, in R.
 - a) (3 marks) Write down the commands that find the **returns** for Stocks A, B and C.
 - b) (2 marks) Write down the commands that estimate the expected returns for Stock B.
 - c) (2 marks) Write down the commands that estimate the covariance between returns of Stocks A and B.
 - d) (3 mark) Write down the commands that find the inverse of estimated returns covariance matrix of Stocks A, B and C.
- 2) (36 marks) Suppose that the estimated expected returns and covariance matrix of returns for the three stocks, and the inverse of the covariance matrix are

$$\hat{\boldsymbol{\mu}} = \begin{pmatrix} 0.01 \\ 0.035 \\ 0.05 \end{pmatrix}, \quad \hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 4 & 1.25 & -0.3 \\ 1.25 & 3 & -2 \\ -0.3 & -2 & 6 \end{pmatrix}, \quad \hat{\boldsymbol{\Sigma}}^{-1} = \begin{pmatrix} 0.29 & -0.144 & -0.0334 \\ -0.144 & 0.499 & 0.159 \\ -0.0334 & 0.159 & 0.218 \end{pmatrix}.$$

- a) (10 marks) Find the optimal portfolio that achieves an expected return of 0.045. What is the volatility of this portfolio?
 - b) (10 marks) Draw the efficient frontier constructed from the three stocks. Indicate Stocks A to C in the graph.
 - c) (4 marks) Find the overall minimum risk portfolio.
 - d) (5 marks) Give the R-commands that find the optimal portfolio that achieves an expected return of 0.025, under the condition that no short-selling is allowed.
 - e) (2 marks) Under the no short-selling constraint, what is the range of target return that a portfolio can achieve? Explain.
 - f) (5 marks) Find the tangency portfolio $\mathbf{w}_M$, and give its expected return and volatility.
- 3) (24 marks) Under the setting of 2), consider two portfolios $\mathbf{w}_p = (0.3, 0.3, 0.4)'$ and $\mathbf{w}_q = (0.4, 0.6, 0)'$.
 - a) (4 marks) Find $\text{Cov}(R_p, R_q)$, where R_p and R_q are the returns of $\mathbf{w}_p$ and $\mathbf{w}_q$, respectively.
 - b) (4 marks) Find the CAPM beta for the portfolios $\mathbf{w}_p$.

- c) (6 marks) Draw the security market line, and plot the points corresponding to the three assets and two portfolios on the same graph.
- d) (4 marks) Find the Sharpe ratio and Treynor ratio of portfolio $\mathbf{w}_q$.
- e) (6 marks) Find the factor risk and non-system risk for portfolio $\mathbf{w}_p$.

Part II: Theory: (30 marks)

- 4) Consider a market with n -assets $\mathbf{R}$ and risk-free rate r . Denote the expected returns and covariance matrix of the returns as $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.
- a) (2 marks) For a portfolio $\mathbf{w} \in \mathbb{R}^n$, express its return variance in terms of $\mathbf{w}$ and $\boldsymbol{\Sigma}$.
 - b) (3 marks) Suppose that the optimal $\mathbf{w}$ under the constraints $\mathbf{w}^T \boldsymbol{\mu} = \mu^*$ and $\mathbf{w}^T \mathbf{1} = 1$ achieves a portfolio variance $\sigma^2(\mu^*)$. State the relationship between $\sigma^2(\mu^*)$ and μ^* .
 - c) (4 marks) Suppose that now we want to maximize the expected return $\mathbf{w}^T \boldsymbol{\mu}$ subject to the constraints that the portfolio variance is $\tilde{\sigma}^2$, and $\mathbf{w}^T \mathbf{1} = 1$. A Lagrangian is formulated as

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}^T \boldsymbol{\mu} - \frac{1}{2\lambda_1} XXX + \lambda_2 YYY, \quad (1)$$

where XXX and YYY are expressions involving $\tilde{\sigma}^2$ and $\mathbf{1}$, respectively. State XXX and YYY .

- d) (4 marks) Show that the optimal portfolio $\mathbf{w}_*$ obtained in c) satisfies $\mathbf{w}_* = \lambda_1 \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} + \lambda_2 \mathbf{1})$.
- e) (4 marks) Show that the optimal portfolio $\mathbf{w}_*$ has an expected return $\tilde{\mu} = \frac{c_{\mu\mu} + \lambda_2 c_{1\mu}}{c_{1\mu} + \lambda_2 c_{11}}$ for some constants $c_{11}, c_{1\mu}, c_{\mu\mu}$. Express these three constants in terms of $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$.
- f) (4 marks) Show that $\tilde{\sigma}^2 = \frac{c_{\mu\mu} + 2\lambda_2 c_{1\mu} + \lambda_2^2 c_{11}}{(c_{1\mu} + \lambda_2 c_{11})^2}$.
- g) (4 marks) Find a quadratic equation that can be used to solve for λ_2 .
- h) (5 marks) Using d)-f), or otherwise, show that the relationship between $\tilde{\sigma}^2$ and $\tilde{\mu}$ is the same as the relationship between $\sigma^2(\mu^*)$ and μ^* in b).

End of paper