

RMSC 4003 Statistical Modelling in Financial Markets

Midterm

9:30-12:15. Tuesday, 17 Oct 2023

Name: _____

Major: _____

Part I: Applications: (70 marks):

Suppose that the risk-free rate is $r = 4\%$. We are interested in constructing an optimal portfolio using three stocks in the market, Stocks A , B and C .

- 1) (10 marks) Suppose that we want to download daily adjusted close prices of Stocks A , B and C from July 2, 1997 to Dec 31, 2022, and the stock numbers of the three stocks are 0609.HK, 0721.HK and 0831.HK, respectively.
 - a) (2 marks) Write down two R-packages that can be used together to download stock data.
 - b) (3 marks) Write down the commands that store the adjusted close price for Stocks A into a vector $\mathbf{PA}$.
 - c) (2 marks) Write down the commands that compute $\mathbf{LA}$, the log-returns of Stock A .
 - d) (1 mark) Write down the commands that output the number of log-returns that we collected for Stock A .
 - e) (2 marks) Write down the commands that estimate the expected log-returns for Stock A .
- 2) (42 marks) Suppose that the estimated expected returns and covariance matrix of returns for the three stocks, and the inverse of the covariance matrix are

$$\hat{\boldsymbol{\mu}} = \begin{pmatrix} -0.01 \\ 0.05 \\ 0.09 \end{pmatrix}, \quad \hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 8 & 2 & -1 \\ 2 & 5 & -4 \\ -1 & -4 & 9 \end{pmatrix}, \quad \hat{\boldsymbol{\Sigma}}^{-1} = \begin{pmatrix} 0.140 & -0.068 & -0.014 \\ -0.068 & 0.343 & 0.145 \\ -0.014 & 0.145 & 0.174 \end{pmatrix}.$$

- a) (10 marks) Find the optimal portfolio that achieves an expected return of 0.07. What is the volatility of this portfolio?
- b) (10 marks) Draw the efficient frontier constructed from the three stocks. Indicate Stocks A to C on the graph.
- c) (4 marks) Find the overall minimum risk portfolio.
- d) (5 marks) Give the R-commands that find the optimal portfolio that achieves an expected return of 0.07, under the condition that no short-selling is allowed.
- e) (5 marks) Find the tangency portfolio $\mathbf{w}_M$, and give its expected return and volatility.
- f) (8 marks) Suppose that Asset D can be replicated by combining Stocks A , B , C with ratios $(0.25, 0.25, 0.5)$. Determine the expected returns and volatilities of all possible portfolios that can be formed by combining Asset D and the tangency portfolio. Draw a curve on the graph in b) to indicate these portfolios.

- 3) (18 marks) Under the setting of 2), suppose that vectors ReM, SMB and HML for the corresponding period are obtained for the three factors in the Fama-French model.
- a) (6 marks) Describe (mathematically, without coding) how SMB and HML are computed.
- b) (4 marks) Suppose that a regression is fit on the excess return of Stock A, $(R_{at} - r_t)$ against the excess return of market portfolio $(R_{Mt} - r_t)$, SMB_t and HML_t , and the results are as follows.

	Intercept	ReM	SMB	HML
Estimate	0.001	0.981	0.422	1.935

Write down the fitted Fama-French three factor model for Stock A. Suppose that $\hat{\mathbf{b}}_A$ denotes the corresponding factor loadings. Write down the values of $\hat{\mathbf{b}}_A$.

- c) (6 marks) Suppose that the above Fama-French model is also fitted to Stocks B and C, and the corresponding factor are denoted as $\hat{\mathbf{b}}_B$ and $\hat{\mathbf{b}}_C$, respectively. Write down the procedure of conducting cross-sectional regression to estimate the factor risk premium λ_0 and $\boldsymbol{\lambda}$ under the Arbitrage Pricing Theory. Define clearly the notations you used in the procedure.
- d) (2 marks) Is the above cross-sectional regression feasible? Explain.

Part II: Theory: (30 marks)

- 1) Consider $n + 1$ risky assets having returns $\mathbf{R} = (R_0, R_1, \dots, R_n)^T$, with expected values $\tilde{\boldsymbol{\mu}} = E(\mathbf{R}) = (\mu_0, \boldsymbol{\mu}^T) = (\mu_0, \mu_1, \dots, \mu_n)^T$ and covariance matrix $\tilde{\Sigma} = \text{Cov}(\mathbf{R}) = (\sigma_{ij})_{0 \leq i, j \leq n}$. Recall that, for a target return μ_* , the optimal portfolio constructed from these assets is given by

$$\mathbf{w}_*(\mu_*) = \tilde{\Delta}^{-1}(\tilde{c}_{11}\tilde{\Sigma}^{-1}\tilde{\boldsymbol{\mu}} - \tilde{c}_{1\mu}\tilde{\Sigma}^{-1}\mathbf{1})\mu_* + \tilde{\Delta}^{-1}(\tilde{c}_{\mu\mu}\tilde{\Sigma}^{-1}\mathbf{1} - \tilde{c}_{1\mu}\tilde{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}). \quad (1)$$

- a) (4 marks) Express $\tilde{c}_{11}, \tilde{c}_{1\mu}, \tilde{c}_{\mu\mu}$ and $\tilde{\Delta}$ in (1) in terms of $\tilde{\boldsymbol{\mu}}, \tilde{\Sigma}$ and any appropriate constants.
- b) (2 marks) Suppose that $\sigma_{00} = v^2$, $\sigma_{0j} = \rho_j v \sqrt{\sigma_{jj}}$, and $\tilde{\Sigma} = \begin{pmatrix} v^2 & v\mathbf{b}^T \\ v\mathbf{b} & \Sigma \end{pmatrix}$. Identify $\mathbf{b}$ and Σ .
- c) (4 marks) Let $a = (1 - \mathbf{b}^T \Sigma^{-1} \mathbf{b})^{-1}$. Show that $\begin{pmatrix} av^{-2} & -av^{-1}\mathbf{b}^T \Sigma^{-1} \\ -av^{-1}\Sigma^{-1}\mathbf{b} & \Sigma^{-1} + a\Sigma^{-1}\mathbf{b}\mathbf{b}^T \Sigma^{-1} \end{pmatrix} \tilde{\Sigma} = I$.
- d) (6 marks) Let $c_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$, $c_{1\mu} = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}$, $c_{1b} = \mathbf{1}^T \Sigma^{-1} \mathbf{b}$, $c_{b\mu} = \mathbf{b}^T \Sigma^{-1} \boldsymbol{\mu}$, $c_{\mu\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$.
- i) (2 marks) Show that $\tilde{c}_{11} = av^{-2}(1 - vc_{1b})^2 + c_{11}$.
- ii) (2 marks) Let $\mu_0 = r$. Show that $\tilde{c}_{1\mu} = av^{-2}(r - vc_{b\mu})(1 - vc_{1b}) + c_{1\mu}$.
- iii) (2 marks) Show that $\tilde{c}_{\mu\mu} = av^{-2}(r - vc_{b\mu})^2 + c_{\mu\mu}$.
- e) (10 marks) Suppose that $v \rightarrow 0$ while other parameters remain fixed.
- i) (2 marks) Show that the leading order term of $\tilde{c}_{11}$ is av^{-2} .
- ii) (2 marks) What is the leading order term of $\tilde{c}_{1\mu}$?
- iii) (2 marks) What is the leading order term of $\tilde{c}_{\mu\mu}$?
- iv) (4 marks) Show that the leading order term of $\tilde{\Delta}$ is $av^{-2}(c_{\mu\mu} - 2rc_{1\mu} + r^2c_{11})$.
- f) (4 marks) Using the above, find $\mathbf{w}_*(\mu_*)$ as $v \rightarrow 0$ while other parameters remain fixed.

End of paper