

RMSC 4003 Statistical Modelling in Financial Markets

Midterm

9:30-12:15. Tuesday, 22 Oct 2024

Name: _____

Major: _____

Part I: Applications: (70 marks):

- 1) (10 marks) Suppose that high frequency data vectors V_A , V_B , and V_C for three stocks A, B, C, respectively, are available in R. Each vector contains prices over n days, with prices at 10 time points included for each day.
 - a) (1 mark) Write down the R codes to find n .
 - b) (3 marks) Write down the R codes to find the daily average prices of the stocks, and store them as PA , PB , and PC respectively.
 - c) (2 marks) Write down the R codes to use daily average prices to compute the **daily log-returns** of the stocks, and store them as LA , LB , and LC respectively.
 - d) (2 marks) Write down the commands that estimate the expected log-returns for Stock A.
 - e) (2 marks) Write down the commands that estimate the variance matrix of the log-returns for the three stocks.
- 2) (40 marks) Suppose that the risk-free rate is $r = 5\%$. We are interested in constructing an optimal portfolio using three risk free asset and three stocks in the market, Stocks A, B and C. Suppose that the estimated expected returns and covariance matrix of returns for the three stocks, and the inverse of the covariance matrix are

$$\hat{\mu} = \begin{pmatrix} 0.02 \\ 0.06 \\ 0.08 \end{pmatrix}, \quad \hat{\Sigma} = \begin{pmatrix} 7 & 2 & -1 \\ 2 & 6 & -3 \\ -1 & -3 & 8 \end{pmatrix}, \quad \hat{\Sigma}^{-1} = \begin{pmatrix} 0.158 & -0.053 & 0 \\ -0.053 & 0.223 & 0.077 \\ 0 & 0.077 & 0.154 \end{pmatrix}.$$

- a) (10 marks) Find the optimal portfolio constructed from only risky assets that achieves an expected return of 0.08. What is the volatility of this portfolio?
- b) (14 marks) Find and draw the efficient frontier constructed from only risky assets. Find the overall minimum variance portfolio (OMVP). Indicate Stocks A to C, and the OMVP on the graph.
- c) (6 marks) Find the tangency portfolio. Find its expected return and volatility, and plot it on the graph in b).
- d) (3 marks) Draw the efficient frontier constructed from both the risk free asset and the three stocks.
- e) (5 marks) Let $\hat{\mu}$ and $\hat{\Sigma}$ be denoted as M and C in R. Give the R-commands that find the optimal portfolio using risky assets only, and achieving an expected return of 0.08, under the condition that no short-selling is allowed.
- f) (2 marks) Let w be a portfolio on the efficient frontier positioned at the top right of the tangency portfolio. What is the sign of its weight allocated on the risk free asset?

- 3) (20 marks) Under the setting of 2), consider two portfolios $\mathbf{w}_p = (0.7, 0.2, 0.1)'$ and $\mathbf{w}_q = (0.8, 0.3, x)'$.
- (2 marks) Find x .
 - (3 marks) Find $\text{Cov}(R_p, R_q)$, where R_p and R_q are the returns of $\mathbf{w}_p$ and $\mathbf{w}_q$, respectively.
 - (3 marks) Find the CAPM beta for the portfolios $\mathbf{w}_p$.
 - (4 marks) Draw the security market line, and plot the points corresponding to the three assets and two portfolios on the same graph.
 - (4 marks) Find the Sharpe ratio and Treynor ratio of portfolio $\mathbf{w}_p$.
 - (4 marks) Suppose that the initial price of the three assets and the market portfolio are \$ (34, 78, 120)', and \$200, respectively, and the initial holding of Stock C is 200 units. Assuming fractional holding is allowed, discuss how to hedge the risk of $\mathbf{w}_p$.

Part II: Theory: (30 marks)

- 4) (22 marks) Consider a market with n -assets $\mathbf{R}$ and risk-free rate r . Denote the expected returns and covariance matrix of the returns as $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.
- (3 marks) Consider a portfolio $\mathbf{w} = (w_1, \dots, w_n)'$ with expected return μ and volatility σ . Show that, on a graph of expected return against volatility, the slope of the line from (σ, μ) and $(0, r)$ is $\frac{\mathbf{w}^T(\boldsymbol{\mu} - r\mathbf{1})}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}}$.
 - (6 marks) Using differentiations, show that an extremum of the slope in a) corresponds to a $\mathbf{w} = \mathbf{w}_M$ satisfying

$$\boldsymbol{\mu} - r\mathbf{1} = \lambda \boldsymbol{\Sigma} \mathbf{w}_M, \quad (1)$$
 for some constant λ .
 - (6 marks) From b) and the fact that $\mathbf{w}^T \mathbf{1} = 1$ for any $\mathbf{w}$, find λ and $\mathbf{w}_M$.
 - (2 marks) From b), show that $\mu_M - r = \lambda \sigma_M^2$.
 - (5 marks) From b) and d), show that $\boldsymbol{\mu} - r\mathbf{1} = \boldsymbol{\Sigma} \mathbf{w}_M \frac{\mu_M - r}{\sigma_M^2}$. From this, deduce the CAPM equation in a complete market.
- 5) (8 marks) Suppose that the return of a portfolio A, denoted as R_a , satisfies $E(R_a) = \mu_a$ and $\text{Var}(R_a) = \sigma_a^2$. Also, let r be the risk-free rate.

- (4 marks) Let P_α be a portfolio constructed from putting a weight α on the risk-free asset and $1 - \alpha$ on A. Show that the curve plotting the expected return against volatility of P_α for various α is a straight line. Find the equation of the straight line in terms of μ_a , r , and σ_a .
- (4 marks) Let B be another portfolio with its return denoted as R_b . Suppose that $E(R_b) = \mu_b$, $\text{Var}(R_b) = \sigma_b^2$, and $\text{Cov}(R_a, R_b) = \sigma_{ab}$. Let Q_α be a portfolio constructed from putting a weight α on B and $1 - \alpha$ on A. Find conditions such that the curve plotting the expected return against volatility of Q_α for various α is a straight line.

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