

RMSC 4003 Statistical Modelling in Financial Markets

Midterm

9:30-12:15. Tuesday, 21 Oct 2025

Name: _____

Major: _____

Part I: Applications: (70 marks):

Suppose that the risk-free rate is $r = 3\%$. We are interested in constructing an optimal portfolio using four stocks in the market, Stocks A , B , C and D .

- 1) (14 marks) Suppose that we want to download daily adjusted close prices of Hong Kong Stocks A , B , C and D from July 21, 2019 to August 31, 2025. The following codes are used:

```
library("tseries")
library("quantmod")
IDs <- matrix(c("0001", "0002", "0003", "0004"))
d0=xxx1; d1=xxx2
Q=xxx3
for (i in c(1:n)) {
  Sec <- xxx4(IDs[i], xxx5)
  Data[,i] <- get.hist.quote(Sec, d0, d1, Q, compression=xxx6)
}
```

- a) (6 marks) Write down the missing commands **xxx1**, **xxx2**, **xxx3**, **xxx4**, **xxx5**, **xxx6**.
- b) (2 marks) It turns out that R reports an error in the codes. What is the problem?
- c) (3 marks) Indicate how to fix the problem in b) in details. (Providing codes is welcome but not necessarily.)
- d) (3 marks) Suppose the problem in b) is fixed. Use `Data` to estimate the expected log-returns and covariance matrix of the log-returns of the four stocks. Call them as `mu.hat` and `sig.hat`, respectively.
- 2) (39 marks) Suppose that the estimated expected returns and covariance matrix of returns for the three stocks, and the inverse of the covariance matrix are

$$\hat{\mu} = \begin{pmatrix} 0.01 \\ 0.03 \\ 0.04 \\ 0.06 \end{pmatrix}, \quad \hat{\Sigma} = \begin{pmatrix} 10 & 5 & 6 & 4 \\ 5 & 8 & 2 & -1 \\ 6 & 2 & 9 & 2 \\ 4 & -1 & 2 & 11 \end{pmatrix}, \quad \hat{\Sigma}^{-1} = \begin{pmatrix} 0.32 & -0.18 & -0.15 & -0.11 \\ -0.18 & 0.23 & 0.049 & 0.076 \\ -0.15 & -0.049 & 0.196 & 0.024 \\ -0.11 & 0.076 & 0.024 & 0.132 \end{pmatrix}.$$

It is given that $\hat{\Sigma}^{-1} \mathbf{1} = (-0.11, 0.18, 0.12, 0.13)'$ and $\hat{\Sigma}^{-1} \hat{\mu} = (-0.014, 0.012, 0.0092, 0.010)'$.

- a) (10 marks) Find the optimal portfolio that achieves an expected return of 0.07. What is the volatility of this portfolio?

- b) (10 marks) Find and draw the efficient frontier constructed from the four stocks. Indicate Stocks A to D on the graph.
- c) (4 marks) Find the overall minimum risk portfolio.
- d) (5 marks) Give the R-commands that find the optimal portfolio that achieves an expected return of 0.07, under the condition that no short-selling is allowed.
- e) (5 marks) Find the tangency portfolio w_M , and give its expected return and volatility.
- f) (5 marks) Explain why each security in the market has a non-zero proportion in the composition of the tangency portfolio.
- 3) (17 marks) Under the setting of 2), suppose that vectors ReM, SMB and HML for the corresponding period are obtained for the three factors in the Fama-French model. Two regressions are fit on the excess return of Stock A, $R_{at} - r_t$ against variables including $R_{Mt} - r_t$, SMB_t and HML_t and the results are as follows.

Fitting 1	Intercept	ReM
Estimate	0.003	1.12

Fitting 2	Intercept	ReM	SMB	HML
Estimate	0.002	0.981	0.422	1.935

Also, the following R codes are executed:

```
print(c(mean(ReM), mean(SMB), mean(HML)))
>[1] 0.033 0.058 0.034
cov(cbind(ReM, SMB, HML))
>
      ReM    SMB    HML
ReM  0.16  0.036  0.17
SMB  0.036  0.012  0.045
HML  0.17  0.045  0.22
```

- a) (4 marks) Describe (analytically, without coding) how HML is computed.
- b) (4 marks) Write down the fitted Fama-French three factor model for Stock A. Suppose that $\hat{b}_A$ denotes the corresponding factor loadings. Write down the value of $\hat{b}_A$.
- c) (2 marks) Find the market price of risk and risk premium in the market.
- d) (3 marks) Based on Fitting 1, find the Jensen index, Sharpe ratio and Treynor ratio of Stock A.
- e) (4 marks) Suppose that a portfolio of value \$ 2M puts weights 0.4 and 0.6 on the risk free asset and Stock A, respectively. If Fitting 2 is employed to describe the return of Stock A, estimate the systematic risk of the portfolio.

Part II: Theory: (30 marks)

Let $\mathbf{R} = (R_1, \dots, R_p)'$ be a p -dimensional random vector containing the returns of p assets. Let $\boldsymbol{\mu} = E(\mathbf{R})$ and $\Sigma = \text{Var}(\mathbf{R})$.

- 4) (24 marks) Roy's criterion determines an optimal portfolio that minimizes the probability that the portfolio return is less than a pre-specified level L , i.e.,

$$\min_{\mathbf{w}} \mathbb{P}(\mathbf{w}^T \mathbf{R} \leq L), \quad \text{subject to } \mathbf{w}^T \mathbf{1} = 1. \quad (1)$$

- a) (4 marks) Assuming that the asset returns follow a multivariate normal distribution, show that

$$\mathbb{P}(\mathbf{w}^T \mathbf{R} \leq L) = \Phi \left(\frac{L - \mathbf{w}^T \boldsymbol{\mu}}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}} \right),$$

for some function Φ . Identify Φ .

- b) (2 marks) Under a), show that the minimization in (1) is equivalent to

$$\min_{\mathbf{w}} \frac{\mathbf{w}^T (L\mathbf{1} - \boldsymbol{\mu})}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}}, \quad \text{subject to } \mathbf{w}^T \mathbf{1} = 1. \quad (2)$$

- c) (6 marks) By optimizing $\frac{\mathbf{w}^T (L\mathbf{1} - \boldsymbol{\mu})}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}}$ without a constraint, show that the optimal $\mathbf{w}$ satisfies $\mathbf{w} = c\Sigma^{-1}(L\mathbf{1} - \boldsymbol{\mu})$ for some constant c .

- d) (5 marks) Using c) and the fact that $\mathbf{w}^T \mathbf{1} = 1$, find c and the optimal $\mathbf{w}$.

- e) (5 marks) Show that for a fixed L , the minimum probability of getting a portfolio return less than L is $\Phi(a\sqrt{b_1 L^2 - b_2 L + b_3})$ for some constants a, b_1, b_2, b_3 . Identify a, b_1, b_2 , and b_3 .

- f) (2 marks) Sketch the graph of the function $\Phi(a\sqrt{b_1 L^2 - b_2 L + b_3})$ in e) against L . Can you identify the L such that the minimum $\Phi(a\sqrt{b_1 L^2 - b_2 L + b_3})$ is achieved?

- 5) (6 marks) Suppose that the determinant of Σ is $\det|\Sigma| = 0$.

- a) (2 marks) Show that there exist some $i \in \{1, \dots, p\}$, $\{j_1, \dots, j_k\} \subset \{1, \dots, p\} \setminus \{i\}$, and real-valued constants $c_{j_1}, \dots, c_{j_k}$ such that with $\tilde{R}_i := \sum_{m=1}^k c_{j_m} R_{j_m}$, we have

$$\text{Cov}(\tilde{R}_i, R_j) = \text{Cov}(R_i, R_j), \quad \text{for every } i \neq j.$$

- b) (2 marks) Is a) equivalent to the statement “There exists some $i \in \{1, \dots, p\}$, $\{j_1, \dots, j_k\} \subset \{1, \dots, p\} \setminus \{i\}$, and real-valued constants $c_{j_1}, \dots, c_{j_k}$ such that $R_i = \sum_{m=1}^k c_{j_m} R_{j_m}$ ”? Explain.

- c) (2 marks) Suppose in addition that $E(\tilde{R}_i) > E(R_i)$. Comment on its effects on the construction of optimal portfolio.

End of paper